Non trivial behavior of the linear response function in phase ordering kinetics*

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Drawing from exact, approximate and numerical results an overview of the properties of the out of equilibrium response function in phase ordering kinetics is presented. Focusing on the zero field cooled magnetization, emphasis is on those features of this quantity which display non trivial behavior when relaxation proceeds by coarsening. Prominent among these is the dimensionality dependence of the scaling exponent a_χ which leads to failure of the connection between static and dynamic properties at the lower dimensionality d_L, where $a_\chi = 0$. We also analyse the mean spherical model as an explicit example of a stochastic unstable system, for which the connection between statics and dynamics fails at all dimensionalities.

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INTRODUCTION

It is a great pleasure to present this talk on the occasion of the 60th birthday of Francesco Guerra. My interest in relaxation phenomena, in fact, initiated quite a few years ago in Salerno where Nelson stochastic mechanics was a subject very much cultivated under the guide of Francesco. I have benefited a lot from his teaching and from the very stimulating atmosphere he was able to create about the applications of stochastic methods to physics in general.

Coming to the subject of this talk, phase ordering [1] is usually regarded as the simplest instance of slow relaxation, quite useful for a first and easy to understand approach to concepts like scaling and aging, which are the hallmarks of glassy behavior [2]. However, next to the similarities there are also fundamental differences which require to keep phase ordering well distinct from the out of equilibrium behavior in glassy systems, both disordered and non disordered. The main source of the differences is the simplicity of the free energy landscape in the case of phase ordering compared to the complexity underlying glassy behavior. This leads to a simple low temperature state for phase ordering systems as opposed to the complexity of replica symmetry breaking for glasses (at least in the mean field picture of spin glasses). An analogous sharp distinction is believed to exist also in the behavior of the out of equilibrium linear response function. As a matter of fact, in the framework where static and dynamic properties are connected [3], systems may be classified on the basis of the fluctuation dissipation relation [4].

In the jargon of slow relaxation, phase ordering is frequently referred to as coarsening. So, if coarsening is associated to simplicity and triviality of behavior what is there to investigate about it? Besides the obvious motivation that the basic, paradigmatic cases need to be thouroughly understood, an additional reason, among others, is that in some cases the existence of complex slow relaxation is identified through the exclusion of coarsening. This requires precise and reliable knowledge of all what goes on when relaxation proceeds by coarsening. An example comes from the long standing controversy about the nature of the low temperature phase of finite dimensional spin glasses. One recent argument in favour of replica symmetry breaking is that the observed behavior of the response function is incompatible with coarsening [3, 4]. This might well be the case. However, for the argument to be sound, the understanding of the out of equilibrium behavior of the response function during phase ordering needs to be up to the level that such a delicate issue demands.

It is the purpose of this talk to present an overview of the accurate investigation of the response function in phase ordering that we have carried out in the last few years. Focusing on the zero field cooled magnetization (ZFC), which is one instance of integrated response function, it will be argued that the response function in phase ordering systems is not as trivial as it is believed to be and, after all, it is not the quantity best suited to highlight the differences between systems with and without replica symmetry breaking. In fact, as we shall see, phase ordering and therefore a replica symmetric low temperature state are compatible with a non trivial ZFC. When this happens there is no

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connection between static and dynamic properties. Phase ordering systems offer examples of two distinct mechanism for the lack of this important feature of slow relaxing systems, the vanishing of the scaling exponent of ZFC and stochastic instability.

PHASE ORDERING

Let us first briefly recall the main features of a phase ordering process. Consider a system, like a ferromagnet, with order parameter (vector or scalar, continous or discrete) $\phi(\vec{x})$ and Hamiltonian $\mathcal{H}[\phi(\vec{x})]$ such that below the critical temperature T_C the structure of the equilibrium state is simple. For example, in the scalar case, there are two pure ordered states connected by inversion symmetry. The form of the Hamiltonian can be taken the simplest compatible with such a structure, like Ginzburg–Landau–Wilson (GLW) for continous spins or the nearest neighbors Ising Hamiltonian for discrete spins.

In the following we will be interested in the space and time dependent correlation function

$$C(\vec{r},t,s) = \langle \phi(\vec{x},t)\phi(\vec{x}',s)\rangle - \langle \phi(\vec{x},t)\rangle\langle \phi(\vec{x}',s)\rangle \tag{1}$$

where the average is taken over initial condition and thermal noise, $\vec{r} = \vec{x} - \vec{x'}$ and $t \geq s \geq 0$ are two times after the quench. The conjugated linear response function is given by

$$R(\vec{r},t,s) = \left.\frac{\delta\langle\phi(\vec{x},t)\rangle}{\delta h(\vec{x'},s)}\right|_{h=0} \tag{2}$$

and ZFC is defined by

$$\chi(\vec{r},t,t_w) = \int_{t_w}^{t} ds R(\vec{r},t,s). \tag{3}$$

Dynamics over phase space: equilibration versus falling out of equilibrium

For a temperature T below T_C, in the thermodynamic limit, the phase space $\Omega = \{[\phi(\vec{x})]\}$ may be regarded as the union of three ergodic components [5] $\Omega = \Omega_+ \cup \Omega_- \cup \Omega_0$, where $\Omega_\pm$ and Ω_0 are the subsets of configurations with magnetization $\lim_{V\to\infty} \frac{1}{V}\int_V d\vec{x}\phi(\vec{x})$ positive, negative and vanishing, respectively. Denoting by $\rho_\pm[\phi(\vec{x})]$ the two broken symmetry pure states, all equilibrium states are the convex linear combinations of $\rho_\pm$. In particular, the Gibbs state is the symmetric mixture $\rho_G[\phi(\vec{x})] = \frac{1}{Z}\exp(-\mathcal{H}[\phi(\vec{x})]/T) = \frac{1}{2}\rho_+[\phi(\vec{x})] + \frac{1}{2}\rho_-[\phi(\vec{x})]$. The $\Omega_\pm$ components are the domains of attraction of the pure states with $\rho_+(\Omega_+) = \rho_-(\Omega_-) = 1$ and Ω_0 is the border in between them, with zero measure in any of the equilibrium states.

When ergodicity is broken, quite different behaviors may arise [5] depending on the initial condition $\rho_0[\phi(\vec{x})] = \rho([\phi(\vec{x})], t = 0)$. Here, we consider the three cases relevant for what follows, assuming that there are not explicit symmetry breaking terms in the equation of motion:

1. *equilibration to a pure state*

   if $\rho_0(\Omega_+) = 1$ or $\rho_0(\Omega_-) = 1$, in the time evolution configurations are sampled from either one of $\Omega_\pm$ and $\rho([\phi(\vec{x})], t)$ equilibrates to the time independent pure state $\rho_\pm[\phi(\vec{x})]$ within the finite relaxation time $t_{eq} \sim \xi^z$, where ξ is the equilibrium correlation lenght and z is the dynamic exponent entering the domain growth law, to be defined shortly. The correlation function is the same in the two ergodic components and, after equilibration, is time translation invariant

   $$C_{st}(\vec{r}, t-s) = \langle \phi(\vec{x},t)\phi(\vec{x}',s)\rangle_\pm - M^2 \tag{4}$$

   where $\langle\phi(\vec{x})\rangle_\pm = \pm M$ is the spontaneous magnetization. For large distances $r \gg \xi$ and time separations $t - s \gg t_{eq}$, the clustering property $\langle \phi(\vec{x},t)\phi(\vec{x}',s)\rangle_\pm = \langle\phi(\vec{x},t)\rangle_\pm\langle\phi(\vec{x}',s)\rangle_\pm$ is obeyed and the correlations decay to zero, as required by ergodicity.

2. *equilibration to the Gibbs state*

   if $\rho_0(\Omega_+) = \rho_0(\Omega_-) = 1/2$, then configurations are sampled evenly from both disjoint components Ω_+ and Ω_-. The probability density $\rho([\phi(\vec{x})], t)$ equilibrates now to the Gibbs state $\rho_G[\phi(\vec{x})]$ with the same relaxation time t_{eq} as in the relaxation to the pure states. Broken ergodicity shows up in the large distance and in the large time properties of the correlation function. After equilibration, one has

$$C_G(\vec{r}, t-s) = C_{st}(\vec{r}, t-s) + M^2 \tag{5}$$

   from which follows that correlations do not vanish asymptotically or that the clustering property is not obeyed

$$\lim_{r\to\infty} C_G(\vec{r}, t-s) = \lim_{(t-s)\to\infty} C_G(\vec{r}, t-s) = M^2. \tag{6}$$

3. *falling out of equilibrium over the border [6, 7]*

   if $\rho_0(\Omega_0) = 1$, for the infinite system $\rho(\Omega_0, t) = 1$ also at any finite time after the quench. Namely, the system does not equilibrate since in any equilibrium state the measure of Ω_0 vanishes. Phase ordering corresponds to this case. In fact, the system is initially prepared in equilibrium at very high temperature (for simplicity $T_I = \infty$) and at the time $t = 0$ is suddenly quenched to a final temperature T below T_C. In the initial state the probability measure over phase space is uniform $\rho_0[\phi(\vec{x})] = 1/|\Omega|$, implying that the initial configuration at $t = 0$ belongs almost certainly to Ω_0, since with a flat measure $|\Omega_0|$ is overwhelmingly larger than $|\Omega_\pm|$.

   The morphology of typical configurations visited as the system moves over Ω_0 is a patchwork of domains of the two competing equilibrium phases, which coarsen as the time goes on. The typical size of domains grows with the power law $L(t) \sim t^{1/z}$, where $z = 2$ (independent of dimensionality) for dynamics with non conserved order parameter [1], as it will be consider here. The sampling of configurations of this type is responsible of the peculiar features of phase ordering. At a given time t_w there remains defined a length $L(t_w)$ such that for space separations $r \ll L(t_w)$ or for time separations $t - t_w \ll t_w$ intra-domains properties are probed. Then, everything goes as in the case 2 of the equilibration to the Gibbs state, ergodicity looks broken and the correlation function obeys Eq. (5). Conversely, for $r \gg L(t_w)$ or $t/t_w \gg 1$, inter-domains properties are probed, ergodicity is restored (as it should be, since evolution takes place within the single ergodic component Ω_0) and eventually the correlation function decays to zero. However, the peculiarity is that if the limit $t_w \to \infty$ is taken before $r \to \infty$, in the space sector ergodicity remains broken giving rise, for instance, to the growth of the Bragg peak in the equal time structure factor.

   According to this picture, the correlation function can be written as the sum of two contributions

$$C(\vec{r}, t, s) = C_{st}(\vec{r}, t-s) + C_{ag}(\vec{r}, t, s) \tag{7}$$

   where the first one is the stationary contribution of Eq. (4) describing equilibrium fluctuations in the pure states and the second one contains all the out of equilibrium information. The latter one is the correlation function of interest in the theory of phase ordering where, in order to isolate it, zero temperature quenches are usually considered as a device to eliminate the stationary component. It is now well established that $C_{ag}(\vec{r}, t, s)$ obeys scaling in the form [8]

$$C_{ag}(\vec{r}, t, s) = \widehat{C}(r/L(s), t/s) \tag{8}$$

   with $\widehat{C}(x, y) = M^2$ for $x < 1$ and $y \sim 1$, while

$$\widehat{C}(r/L(s), t/s) \sim (t/s)^{-\lambda/z} h(r/L(s)) \tag{9}$$

   for large time separation [1], where λ is the Fisher–Huse exponent.

ZERO FIELD COOLED MAGNETIZATION

Let us next consider what happens when a time independent external field $h(\vec{x})$ is switched on at the time t_w. To linear order the expectation value of the order parameter at the time t is given by

$$\langle\phi(\vec{x},t)\rangle_h = \langle\phi(\vec{x},t)\rangle_0 + \int d\vec{x'}\chi(\vec{x}-\vec{x'},t,t_w)h(\vec{x'}) \tag{10}$$

and if $h(\vec{x})$ is random with expecations $\overline{h(\vec{x})} = 0$, $\overline{h(\vec{x})h(\vec{x'})} = h_0^2\delta(\vec{x}-\vec{x'})$ then one has

$$\chi(\vec{x}-\vec{y},t,t_w) = h_0^{-2}\overline{\langle\phi(\vec{x},t)\rangle_h h(\vec{y})}. \tag{11}$$

Namely, ZFC is the correlation at the time t of the order parameter with the random external field.
Going to the three processes considered above

1. after equilibration in the pure state has occurred and the stationary regime has been entered, the order parameter correlates with the external field via the equilibrium thermal fluctuations. The fluctuation dissipation theorem (FDT) is obeyed

$$\chi_{st}(\vec{r},t-t_w) = \frac{1}{T}\left[C_{st}(\vec{r},t-s=0) - C_{st}(\vec{r},t-t_w)\right] \tag{12}$$

and since $C_{st}(\vec{r},t-t_w)$ decays to zero for $t-t_w > t_{eq}$, over the same time scale $\chi_{st}(\vec{r},t-t_w)$ saturates to

$$\lim_{t\to\infty}\chi_{st}(\vec{r},t-t_w) = \chi_{eq}(\vec{r}) = \frac{1}{T}C_{eq}(\vec{r}) \tag{13}$$

which is the susceptibility computed in the final equilibrium state $\rho_\pm[\phi(\vec{x})]$.

2. As far as ZFC is concerned, there is no difference between the relaxation to the mixed Gibbs state and the relaxation to a pure state. Hence, FDT is satisfied and can be written both in terms of C_{st} or C_G since, as Eq. (5) shows, they differ by a constant.

3. In the phase ordering process the system stays out of equilibrium, so it useful to write ZFC as the sum of two contributions [9]

$$\chi(\vec{r},t,t_w) = \chi_{st}(\vec{r},t-t_w) + \chi_{ag}(\vec{r},t,t_w) \tag{14}$$

where $\chi_{st}(\vec{r},t-t_w)$ satisfies Eq. (12) and $\chi_{ag}(\vec{r},t,t_w)$ represents the additional out of equilibrium response. In connection with this latter contribution there are two basic questions

i) how does it behave with time

ii) what is the relation between χ_{ag} and C_{ag}, if any.

Scaling hypothesis

Since ZFC measures the growth of correlation between the order parameter and the external field, the first question raised above addresses the problem of an out of equilibrium mechanism for this correlation, in addition to the thermal fluctuations accounting for χ_{st}. Restricting attention from now on, for simplicity, to the case of coincident points ($\vec{r}=0$) and dropping therefore the space dependence, the starting point for the answer is the assumption of a scaling form

$$\chi_{ag}(t,t_w) \sim t_w^{-a_\chi} \widehat{\chi}_{ag}(t/t_w) \tag{15}$$

which is the counterpart of Eq. (8) for the correlation function.

The next step is to make statements on the exponent a_χ and on the scaling function $\widehat{\chi}_{ag}(x)$. There exists in the literature an estimate of a_χ based on simple reasoning. What makes phase ordering different from relaxation in the pure or in the Gibbs state is the existence of defects. The simplest assumption is that $\chi_{ag}(t,t_w)$ is proportional to the density of defects [3, 4, 10]. This implies

$$a_\chi = \delta \tag{16}$$

where the exponent δ regulates the time dependence of the density of defects $\rho_{defect}(t) \sim L(t)^{-n} \sim t^{-\delta}$, namely

$$\delta = n/z \tag{17}$$

with $n = 1$ for scalar and $n = 2$ for vector order parameter [1].

According to this argument a_χ should be independent of dimensionality. This conclusion is not corroborated by the available exact, approximate and numerical results. On the basis of exact analytical solutions for the $d = 1$ Ising model [11, 12] and for the large N model [13], approximate analytical results based on the gaussian auxiliary field (GAF) approximation [14, 15] and numerical results from simulations of the Ising model [15, 16, 17] with $d = 2, 3, 4$, the following general formula for a_χ has been obtained

$$a_\chi = \begin{cases} \delta \left(\frac{d-d_L}{d_U-d_L} \right) & \text{for} \quad d < d_U \\ \delta \quad \text{with log corrections for} & d = d_U \\ \delta \quad \text{for} \quad d > d_U \end{cases} \tag{18}$$

where d_L and $d_U > d_L$ do depend on the system in the following way

- d_L is the dimensionality where $a_\chi = 0$. In the Ising model $d_L = 1$, while in the large N model $d_L = 2$. The speculation is that in general $d_L = 1$ for systems with discrete symmetry and $d_L = 2$ for systems with continous symmetry, therefore suggesting that d_L coincides with the lower critical dimensionality of equilibrium critical phenomena, although the reasons for this identification are far from clear.

- d_U is a value of the dimensionality specific of ZFC and separating $d < d_U$, where a_χ depends on d, from $d > d_U$ where a_χ is independent of dimensionality and Eq. (16) holds. The existence of d_U is due [17] to a mechanism, i.e. the existence of a dangerous irrelevant variable, quite similar (including logarithmic corrections) to the one leading to the breaking of hyperscaling above the upper critical dimensionality in static critical phenomena. However, d_U cannot be identified with the upper critical dimensionality since we have found, so far, $d_U = 3$ in the Ising model and $d_U = 4$ in the large N model. In the scalar case it may be argued [18] that d_U coincides with the dimensionality $d_R = 3$ such that interfaces do roughen for $d \leq d_R$ and do not for $d > d_R$. A general criterion for establishing the value of d_U is not yet known.

The validity of Eq. (15) with a_χ given by Eq. (18) has been checked, in addition to the cases where analytical results are available, with very good accuracy in the simulations of the Ising model [17]. The values of δ, d_L and d_U obtained for the different systems are collected in Table I and the behavior of a_χ as dimensionality is varied is displayed in Fig.1.

	Ising	GAF	$N = \infty$
δ	1/2	1/2	1
d_L	1	1	2
d_χ	3	2	4

TABLE I: Parameters entering Eq.(18) in various models.

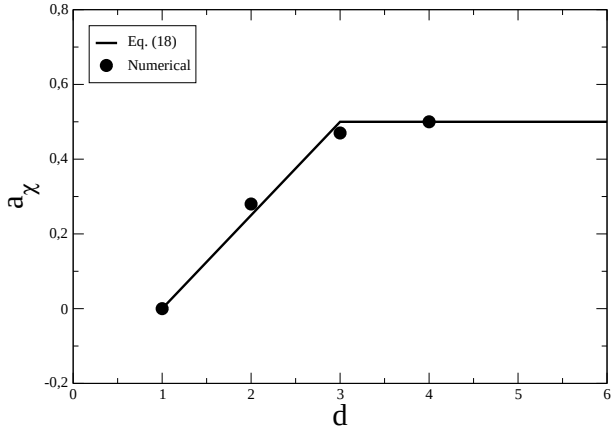

FIG. 1: Exponent a_χ in the Ising model at various dimensionalities. The continous line represents Eq. (18), while the dots are the values from the exact solution[11] of the model at $d = 1$ and from simulations[17] at $d = 2, 3, 4$.

At this point, a comment is in order. From Eqs. (14) and (15) follows that the existence of out of equilibrium degrees of freedom, or defects, generates the aging contribution to ZFC. However, when $a_\chi > 0$ this is bound to disappear as time becomes large, either $t_w \to \infty$ or $t/t_w \to \infty$. The vanishing of this contribution is fastest for $d > d_U$, where $a_\chi = \delta$, and becomes slower and slower below d_U, as d_L is approached. Nonetheless, as long as $a_\chi > 0$, eventually χ_{ag} disappears and Eq. (13) holds also for phase ordering processes for all $d > d_L$. This is no longer true for $d \leq d_L$ where $a_\chi \leq 0$ and, consequently, χ_{ag} gives a contribution to the response which persists also in the asymptotic time region leading to

$$\lim_{t\to\infty} \chi(t, t_w) > \chi_{eq} \tag{19}$$

where χ_{eq} stands for $\chi_{eq}(\vec{r} = 0)$. In the following we will consider $d \geq d_L$.

Fluctuation dissipation relation

Let us now come to the second question. A very useful tool for the study of slow relaxation phenomena has been introduced by Cugliandolo and Kurchan [19] through the off equilibrium fluctuation dissipation relation. This can be introduced in the following way. Given that $C(t, t_w)$ is a monotonously decreasing function of t, for fixed t_w it is possible to invert it and write

$$\chi(t, t_w) = \widetilde{\chi}(C(t, t_w), t_w). \tag{20}$$

Then, if for a fixed value of $C(t, t_w)$ there exists the limit

$$\lim_{t_w\to\infty} \widetilde{\chi}(C, t_w) = S(C) \tag{21}$$

the function $S(C)$ gives the fluctuation dissipation relation. In the particular case of equilibrium dynamics, FDT is recovered and $S(C) = [C(0) - C]/T$. Originally introduced in the study of the low temperature phase of spin glass mean-field models, the fluctuation dissipation relation has been found in many other instances of slow relaxation [20].

Now, in order to search for $S(C)$ in the case of phase ordering, let us set $\vec{r} = 0$ in Eq. (8) and let us eliminate t/t_w between $\widehat{\chi}_{ag}$ and C_{ag} obtaining

$$\chi_{ag}(t, t_w) \sim t_w^{-a_\chi} \widetilde{\chi}_{ag}(C_{ag}). \tag{22}$$

Then, from Eqs. (14,12,22) one can write the general relation

$$\chi(t,t_w) = \frac{1}{T}\left[C_{st}(0) - C_{st}(t-t_w)\right] + t_w^{-a_\chi}\widetilde{\chi}_{ag}(C_{ag}). \tag{23}$$

Using the identity $[C_{st}(0) - C_{st}(t-t_w)] = \left[C_{st}(0) + M^2 - C_{st}(t-t_w) - M^2\right]$ and considering that in the time interval where $C_{st}(t-t_w) \neq 0$, i.e. for short times, one can replace $C_{ag}(t/t_w)$ with M^2 or equivalently $C_{st}(t-t_w)+M^2 = C(t,t_w)$, the above equation can be rewritten as

$$\chi(t,t_w) = \widetilde{\chi}_{st}(C) + t_w^{-a_\chi}\widetilde{\chi}_{ag}(C_{ag}) \tag{24}$$

where the function $\widetilde{\chi}_{st}(C)$ is defined by

$$T\widetilde{\chi}_{st}(C) = \begin{cases} [C(0) - C(t,t_w)] & \text{for} \quad M^2 \leq C \leq C(0) \\ \left[C(0) - M^2\right] & \text{for} \quad C < M^2. \end{cases} \tag{25}$$

Therefore, from Eq. (24) we have, first of all, that for phase ordering systems the fluctuation dissipation relation exists only if $a_\chi \geq 0$ (i.e. for $d \geq d_L$) and that for $a_\chi > 0$

$$S(C) = \widetilde{\chi}_{st}(C). \tag{26}$$

For $a_\chi = 0$ a little more care is needed. Equation (24) yields $\chi(t,t_w) = \widetilde{\chi}_{st}(C) + \widetilde{\chi}_{ag}(C_{ag})$. Recalling that $a_\chi = 0$ occurs at $d = d_L$, which coincides with the lower critical dimensionality, in order to have a phase ordering process a quench to $T = 0$ is required. This, in turn, implies $C_{st}(t,s) = 0$ and $C_{ag}(t,s) = C(t,s)$. Therefore, using Eq. (25) we have

$$S(C) = \chi^*_{eq} + \widetilde{\chi}_{ag}(C) \tag{27}$$

where $\chi^*_{eq} = \lim_{T\to 0}[C(0) - M^2]/T$ is the $T = 0$ equilibrium susceptibility, which vanishes for hard spins while is different from zero for soft spins. Physical implications of these results are discussed in the next section.

STATICS FROM DYNAMICS

One of the main reasons of interest in the fluctuation dissipation relation is that it may provide a link between static and dynamic properties. This was first found by Cugliandolo and Kurchan [19] for mean-field spin glasses and then established in general by Franz *et al.* [3] for slowly relaxing systems.

Let us first introduce the overlap probability function $\widetilde{P}(q)$ in the equilibrium state obtained when the perturbation responsible of $\chi(t,t_w)$ is switched off. The question is how is $\widetilde{P}(q)$ related to the unperturbed overlap function $P(q)$. If $P(q) = \widetilde{P}(q)$ the system is stochastically stable [21]. A milder statement of stochastic stability is that $\widetilde{P}(q)$ coincides with $P(q)$ up to the effects of a global symmetry which might be removed by the perturbation. In the Ising case, where the perturbation breaks the up-down symmetry, defining

$$\widehat{P}(q) = 2\theta(q)P(q) \tag{28}$$

the system is stochastically stable in the sense that $\widetilde{P}(q) = \widehat{P}(q)$. If the system is not stochastically stable, $\widetilde{P}(q)$ is not related neither to $P(q)$ nor to $\widehat{P}(q)$. As we shall see, this is the case of the mean spherical model.

The statement is that

1. if $S(C)$ exists
2. if $\lim_{t\to\infty}\chi(t,t_w) = \chi_{eq}$
3. if the system is stochastically stable

then one has

$$-T\frac{d^2S(C)}{dC^2}\bigg|_{C=q} = \widetilde{P}(q). \tag{29}$$

We may now check if this chain of connections applies to phase ordering. In replica symmetric low temperature states, as for instance in ferromagnetic systems, the overlap function is always trivial and we have

$$P(q) = \frac{1}{2}\left[\delta(q-M^2) + \delta(q+M^2)\right] \tag{30}$$

with

$$\widetilde{P}(q) = \widehat{P}(q) = \delta(q-M^2). \tag{31}$$

Computing the derivative in the left hand side of Eq. (29) and using Eqs. (26) and (27), for $d > d_L$ we find

$$-T\frac{d^2S(C)}{dC^2}\bigg|_{C=q} = \delta(q-M^2). \tag{32}$$

Therefore, Eq. (29) is satisfied.

Failure at $d = d_L$

Let us, next, go to the case $d = d_L$. For the sake of definiteness, we consider the case of the Ising model with $d = 1$. In order to make compatible the two requirements of having an ordered equilibrium state and a well defined linear response function, instead of taking the $T \to 0$ limit it is necessary to take the limit of an infinite ferromagnetic coupling [11]. Then, $P(q)$ and $\widetilde{P}(q)$ are given by Eqs. (30) and (31) with $M^2 = 1$ at all temperatures. On the other hand, for any T we also have [11]

$$T\widetilde{\chi}_{ag}(C) = \frac{\sqrt{2}}{\pi}\arctan\left[\sqrt{2}\cot\left(\frac{\pi}{2}C\right)\right]. \tag{33}$$

This gives

$$-T\frac{d^2S(C)}{dC^2}\bigg|_{C=q} = \frac{\pi\cos(\pi q/2)\sin(\pi q/2)}{[2-\sin(\pi q/2)]^2}. \tag{34}$$

Hence, it is clear that Eq. (29) is not verified. The reason is that the second of the above conditions required for establishing the connection is not satisfied. In fact, from Eqs. (24) and (33), keeping in mind that the limits $t \to \infty$ and $C \to 0$ are equivalent, we have

$$\lim_{t\to\infty} T\chi(t,t_w) = 1/\sqrt{2} \tag{35}$$

which is responsible of Eq. (19) and, therefore, of the violation of condition (2) above Eq. (29), since in this case $\chi_{eq} = 0$.

Failure by stochastic instability

An interesting example [23], where statics cannot be reconstructed from dynamics because the third requirement of stochastic stability is not satisfied, comes from the spherical model. More precisely, one must consider in parallel the original version of the spherical model (SM) of Berlin and Kac [24] and the mean spherical model (MSM) introduced

by Lewis and Wannier [25], with the spherical constraint treated in the mean. These two models are equivalent above but not below T_C [26]. The low temperature states are quite different, with a bimodal order parameter probability distribution in the SM case and a gaussian distribution centered in the origin in the MSM case. The corresponding overlap functions are also very different [23]. Considering, for simplicity, $T = 0$ one has

$$P(q) = \begin{cases} \frac{1}{2}\left[\delta(q - M^2) + \delta(q + M^2)\right] & \text{for SM} \\ \frac{1}{\pi M^2} K_0(|q|/M^2) & \text{for MSM} \end{cases} \tag{36}$$

where K_0 is a Bessel function of imaginary argument (Fig.2).

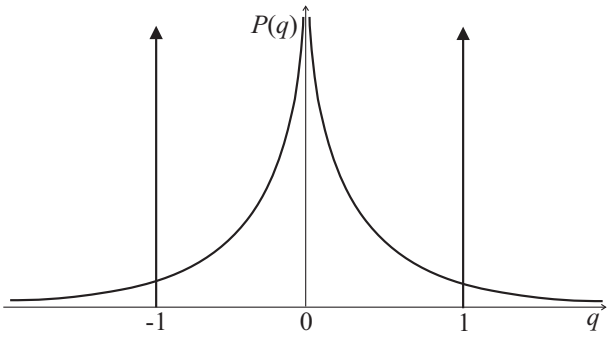

FIG. 2: Overlap distribution for mean spherical model with $M^2 = 1$. The arrows represent the δ functions of the overlap distribution for the spherical model.

However, after switching off an external field, one finds for *both* models $\widetilde{P}(q) = \delta(q - M^2)$. This means that stochastic stability holds for SM but not for MSM.

On the other hand, the relaxation properties are the same in the two models, both above and below T_C if the thermodynamic limit is taken before the $t \to \infty$ limit [23]. Then, the linear response function is the same for both models and obeys Eq. (24) with a_χ given by Eq. (18), where δ, d_L and d_U are the same as for the large N model (Table I). Hence, we have that although Eq. (29) is satisfied for both models, nonetheless statics and dynamics are connected only in the SM case, where $\widetilde{P}(q) = \widehat{P}(q)$. Instead, this is not possible in the MSM case where $\widetilde{P}(q) \neq \widehat{P}(q)$.

CONCLUSIONS

We have shown that although the low temperature equilibrium state in phase ordering systems is simple, ZFC in quenches below T_C displays non trivial features. Summarising, these are

- The aging component $\chi_{ag}(t, t_w)$ obeys the scaling form (15) with an exponent a_χ dependent on dimensionality according to Eq. (18).
- The fluctuation dissipation relation $S(C)$ is trivial, in the sense that it is consistent with a replica symmetric equilibrium state, for $d > d_L$. However, as $d \to d_L$ from above the non trivial contribution due to $\chi_{ag}(t, t_w)$ persists for longer and longer times as $a_\chi \to 0$.
- For $d = d_L$ the fluctuation dissipation relation is non trivial, since the aging contribution does not disappear asymptotically. From this follows i) that there is no connection between static and dynamic properties and ii) that on the basis of the behavior of ZFC is not possible to establish that the equilibrium state is replica symmetric.
- The mean spherical model provides an explicit example of a system whose low temperature equilibrium state is not stochastically stable, producing the failure of the link between static and dynamic properties for all dimensionalities.
- Extensive numerical simulations for ferromagnetic systems with scalar, vector, conserved and non conserved order parameter, at various space dimensionalities are under way. Preliminary results do show that the behavior of the response function above illustrated might well be generic for phase ordering systems [18].

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