Disorder-induced rounding of the phase transition
in the large-$q$-state Potts model

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Motivations

- **Critical Properites of Disordered Systems**
- **Effects of Disorder on First Order Phase Transitions**
- \( \Rightarrow \) **Random Bond Potts Model (RBPM), especially in the large-\( q \)-limit, is a perfect ground to analyze this aspect**
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- Critical properites of disordered systems
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- \( \Rightarrow \) Random Bond Potts Model (RBPM), especially in the large-\( q \)-limit, is a perfect ground to analyze this aspect

Outline

- Potts model in the random cluster representation
- Introducing disorder
- Results and perspectives
The q-state Potts model is defined by:

\[
Z = \sum_{\{\sigma\}} q^{-\beta \mathcal{H}(\{\sigma\})}
\]

where \( \mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \delta(\sigma_i, \sigma_j) \) for \( \sigma_i = 0, 1, \ldots, q - 1 \). The \( J_{ij} \) are FM random couplings.

The random cluster representation is:

\[
Z = \sum_{G \subseteq E} q^{c(G)} \prod_{ij \in G} \nu_{ij} \quad \nu_{ij} = e^{\beta J_{ij}} - 1
\]
$q=3$

0 1 2

RBPM in the large-$q$ limit

M.T. Mercaldo, J-C. Anglès d'Auriac, F. Iglói
\[ c(G) = 10 + 12 = 22 \]
$$c(G) \prod_{e} (e^{\beta J_e} - 1)$$
Properties of homogeneous Potts model

The properties of the homogeneous Potts model can be studied in various limits. In the large-$q$ limit, the Potts model exhibits a first-order phase transition (PT) at a critical point $q_c$. The critical behavior in two dimensions ($2D$) is given by:

- For $q > q_c$, the first-order PT is strongly $1^{st}$ order.
- In the $q!1^{st}$ limit, the PT is strongly $1^{st}$ order.

In systems with disorder, the Harris criterion is used to determine the relevance of disorder:

- If $P > 0$, disorder is relevant.
- If $P < 0$, disorder is irrelevant.

Relevant disorder affects the conventional random fixed-point (FP) behavior, while infinite randomness FP (IRFP) builds up disorder that grows without limits.

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**Congresso del Dipartimento di Fisica “E.R. Caianiello”— 19-20 Aprile 2004, Salerno.**

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\[ q < q_c \Rightarrow 2^{nd} \text{ order PT} \]
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in 2D: \( q_c = 4 \) (exact result)

in the \( q \to \infty \) limit
the PT is strongly 1\(^{st}\) order
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Systems with Disorder

Continuous PT

\[
\begin{align*}
\text{HARRIS criterion} \\
\alpha_P > 0 & \text{ disorder is relevant} \\
\alpha_P < 0 & \text{ disorder is irrelevant}
\end{align*}
\]

1^{st} \text{ order PT}

\[\notin \text{ criterion} \]

it is only known that DISORDER

will SOFTEN the transition


RBPM in the large-\(q\) limit

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$1^{st}$ order PT

$\not{\exists}$ criterion

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Relevant disorder

Conventional Random FP (different values of critical exponents)

Infinite Randomness FP (IRFP) (disorder grows without limits)
\[ T \to T' = T \ln q \quad f(T) \to \frac{f(T')}{\ln q} \]

\[ Z = \sum_{G \subseteq E} q^{c(G)} \prod_{i,j \in G} \left[ q^{\beta J_{ij}} - 1 \right] \]

\[ Z = \sum_{G \subseteq E} q^{\phi(G)} \]

\[ \phi(G) = c(G) + \beta \sum_{i,j \in G} J_{ij} \]
$q \to \infty$ limit

\[
T \to T' = T \ln q \\
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\]

\[
Z = \sum_{G \subseteq E} q^{c(G)} \prod_{i,j \in G} [q^{\beta J_{ij}} - 1]
\]

\[
\downarrow_{q \to \infty}
\]

\[
Z = \sum_{G \subseteq E} q^{\phi(G)}
\]

\[
\phi(G) = c(G) + \beta \sum_{i,j \in G} J_{ij}
\]

\[
Z = n_0 q^{\phi^*}(1 + \ldots)
\]

where \(\phi^* = \max_G \phi(G)\)

and \(\phi^* = -\beta N f\)
All information about the RBPM in the large-$q$ limit is contained in the **OPTIMAL SET $G^*$**
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Thermal properties are calculated from $\phi^*$
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**THERMAL PROPERTIES ARE CALCULATED FROM** \(\phi^*\)

- free energy, internal energy, specific heat,...
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- **Thermal Properties are Calculated from \( \phi^* \)**
  - free energy, internal energy, specific heat, ...

- **Magnetization and Correlation Functions are Obtained from the Geometrical Structure of \( G^* \)**
  - \( C(r) \), average correlation function, is related to the distribution of clusters
  - \( m \), magnetization, is the fraction of sites in the infinite cluster
  - \( \xi \), correlation length, is the average size of the clusters
One has to find the maximizer over the 2-j E possible configuration! A supermodular function \( (A + B) \geq (A \cdot B) \) for \( A;B \in E \) theorem of discrete math a combinatorial optimization method to maximize it in polynomial time for (G) of the Potts model as a specific algorithm has been formulated.

\( d = 2 \), \( L = 512 \) \( 2^{524288} \).
One has to find the max over the $2^{|E|}$ possible configurations!
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- $\phi^*$ is a supermodular function $\Rightarrow \phi(A) + \phi(B) \leq \phi(A \cup B) + \phi(A \cap B)$ $\forall A, B \in E$
maximize $\phi^*$

- One has to find the max over the $2^{|E|}$ possible configuration!

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- Theorem of discrete math $\Rightarrow$ $\exists$ a combinatorial optimization method to maximize it in polynomial time
● One has to find the max over the $2^{|E|}$ possible configuration!

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Angles d’Auriac et al. JPA35, 6973 (2002)
• One has to find the max over the $2^{|E|}$ possible configurations!

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Angles d’Auriac et al. JPA35, 6973 (2002)

\[ d = 2 \quad L = 512 \quad \Rightarrow \quad 2^{524288} \sim 2.6 \cdot 10^{157826} \]
RBPM in the large-$q$ limit

M.T. Mercaldo, J-C. Anglès d'Auriac, F. Iglói
\[
P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right)
\]

\[T = 1.200\]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \]

\[ T = 1.166 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \]

\[ T = 1.066 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 1.050 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \] \quad T = 1.042
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 1.033 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 1.025 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 1.016 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 1.008 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 1.000 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 0.992 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 0.983 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 0.967 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 0.9416 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 0.667 \]
\[ P(J) = \frac{1}{2} \delta \left( J - \frac{1}{6} \right) + \frac{1}{2} \delta \left( J - \frac{5}{6} \right) \quad T = 0.500 \]
IN2DDISORDERDESTROYPHASEOC

itsoftensthe1storderPTintoa2ndorderPT

IN3DWEAKDISORDERDOESNOTDISTROYPHASEOC

ie.disorderhastobestrongenoughtosoftenthePTintobreakinglenght2ndorderabove

orderedphaseT/d

breakinglenght2ndorderbelow

orderedphaseT/d

breakinglenght1storderabove

1storderbelowbreakinglenght

Inafinitesizesystemweakdisorderfluctuationcouldnotbesufficienttobreakphasecoexistence.
In 2D Disorder destroy Phase Coexistence $\Rightarrow$ it softens the $1^{st}$ order PT into a $2^{nd}$ order PT.
**In 2D Disorder destroy Phase Coexistence** ⇒ it softens the $1^{st}$ order PT into a $2^{nd}$ order PT

**In 3D Weak Disorder does not destroy Phase Coexistence** i.e. disorder has to be strong enough to soften the PT into $2^{nd}$ order PT
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\[
\begin{align*}
\delta &
\begin{array}{c}
\text{2D} \\
\text{ordered phase}
\end{array}
\end{align*}
\]

2nd order above breaking length

2nd order below breaking length

\[
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2nd order

1st order above breaking length

1st order below breaking length

In 2D Disorder destroy Phase Coexistence → it softens the 1st order PT into a 2nd order PT

In 3D Weak Disorder does not destroy Phase Coexistence i.e. disorder has to be strong enough to soften the PT into 2nd order PT

In a finite size system weak disorder fluctuation could not be sufficient to break phase coexistence
Through extreme value statistics one can estimate the **breaking length scale** $L \sim \exp[(1/\delta)^2]$

[the finite length scale $L$ at which breaking of phase coexistence take place]

strength of disorder $\delta = \Delta/J$
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Through extreme value statistics one can estimate the **breaking length scale** \( L \sim \exp[(1/\delta)^2] \)

[the finite length scale \( L \) at which breaking of phase coexistence take place]

\[
\text{strength of disorder } \delta = \frac{\Delta}{J}
\]
Free Energy:  \[ F = c(G^*)T - \sum_{ij \in G^*} J_{ij} \]
Internal Energy:  \[ E = -\sum_{ij \in G^*} J_{ij} \]

- For a given sample \( E \) is a piecewise constant function of temperature \( \Rightarrow \) it shows discontinuities
- The average over disorder generally smears out discontinuities
- The behavior of averaged quantities is different for the discrete and the continuous distributions
At the critical point the largest cluster of $G^*$ is a fractal and its mass $M \sim L^{d_f}$

$$d_f = d - \frac{\beta}{\nu}$$

$$d_f = \frac{5 + \sqrt{5}}{4}$$

According to scaling theory, cumulative distribution of the mass of the cluster

$$R(M, L) = M^{-\tau} \tilde{R}(M/L^{d_f})$$
Results in 2D

\[ \alpha = 0, \ \beta = \frac{3 - \sqrt{5}}{4}, \ \nu = 1 \quad \text{as for the RTIM} \Rightarrow \text{IRFP} \]

We can argue that the RTIM is the Hamiltonian version of the 2D RBPM in the large-q limit

Ref.: Mercaldo, Anglès d’Auriac, Iglói, PRE 69, 0461xx (2004); Anglès d’Auriac, Iglói PRL 90, 190601 (2003)
Conclusions

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**Questions in 3D**

- Is the transition line \( T_c/d = 1 \) for \( \delta \ll 1 \) ?

- Is \( \delta = 1/2 \) the tricritical point ?

- Does the critical line depend on the disorder distribution ?

Ref.: Anglès d’Auriac, Iglói, Mercaldo work in progress
Conclusions

★ RESULTS IN 2D

⇒ $\alpha = 0$, $\beta = \frac{3 - \sqrt{5}}{4}$, $\nu = 1$ as for the RTIM $\Rightarrow$ IRFP !

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★ RELATED PROBLEM

⇒ Critical Properties of Quantum Potts model

Ref.: Mercaldo, De Cesare, work in progress