Critical behavior of Random Bond Potts model in the limit of infinite number of states

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Motivations

- CRITICAL PROPERTIES OF DISORDERED SYSTEMS
- EFFECTS OF DISORDER ON FIRST ORDER PHASE TRANSITIONS
- $\Rightarrow$ RANDOM BOND POTTS MODEL (RBPM), ESPECIALLY IN THE LARGE-q-LIMIT, IS A PERFECT GROUND TO ANALIZE THIS ASPECT

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Outline

- POTTS MODEL IN THE RANDOM CLUSTER REPRESENTATION
- INTRODUCING DISORDER
- RESULTS AND PERSPECTIVES

q-state Potts model

$$Z \equiv \sum_{\{\sigma\}} e^{-\beta\mathcal{H}(\{\sigma\})}$$

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\delta(\sigma_i,\sigma_j) \qquad \sigma_i = 0, 1, \cdots, q-1$$

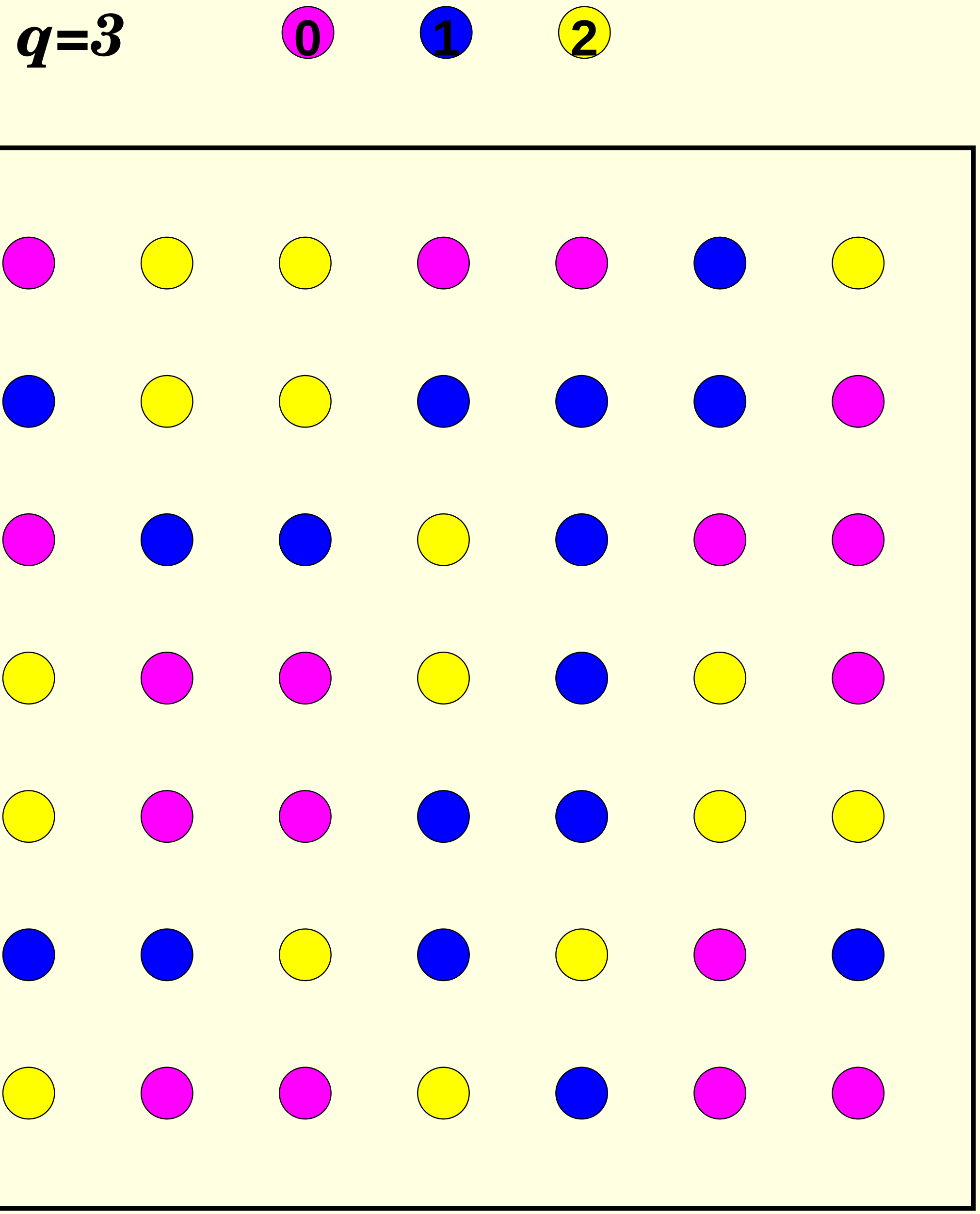
$$\Downarrow$$

Random cluster representation

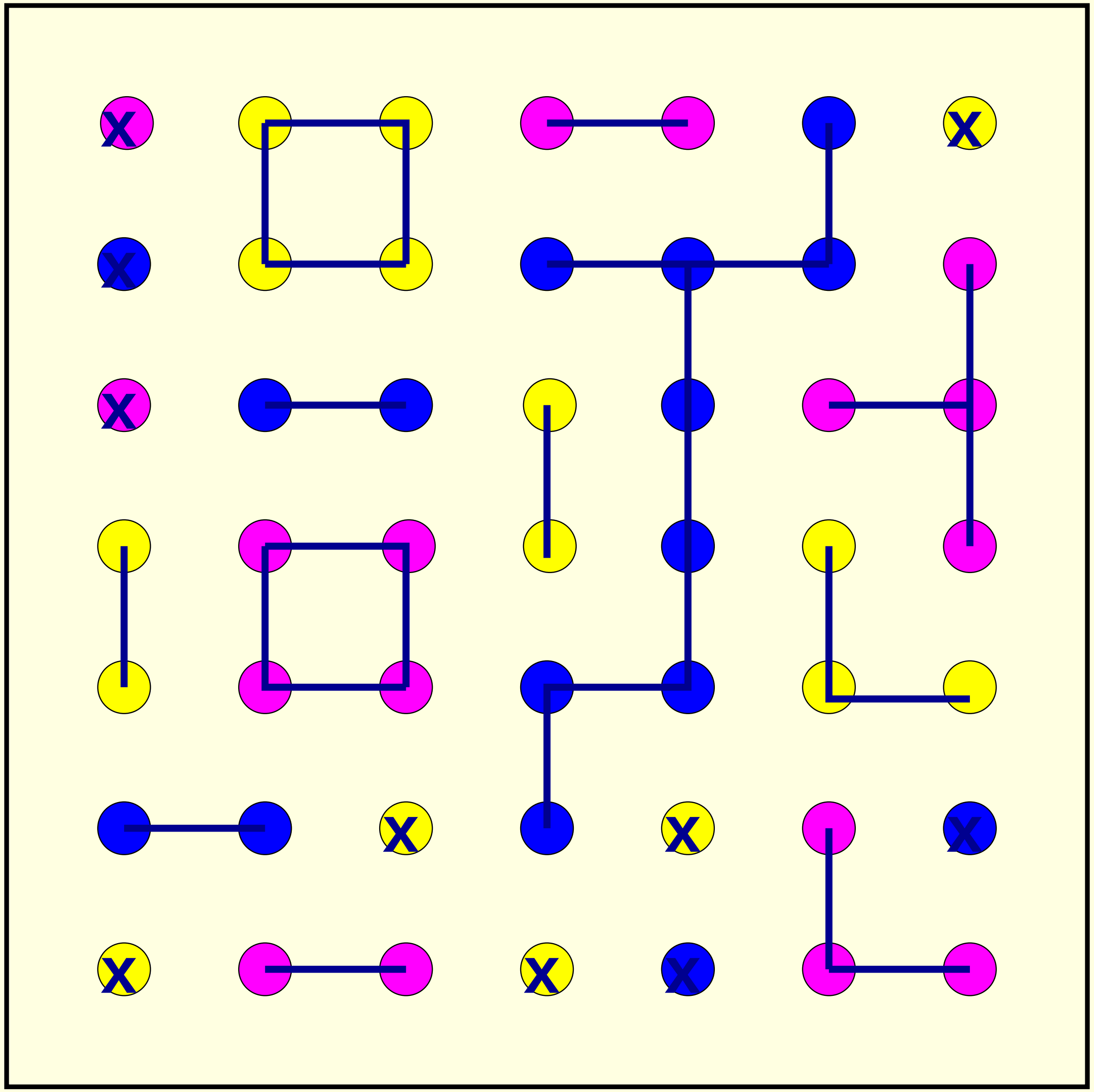
$$Z = \sum_{G\subseteq E} q^{c(G)} \prod_{ij\in G} \nu_{ij} \qquad \nu_{ij} = e^{\beta J_{ij}} - 1$$

About disorder

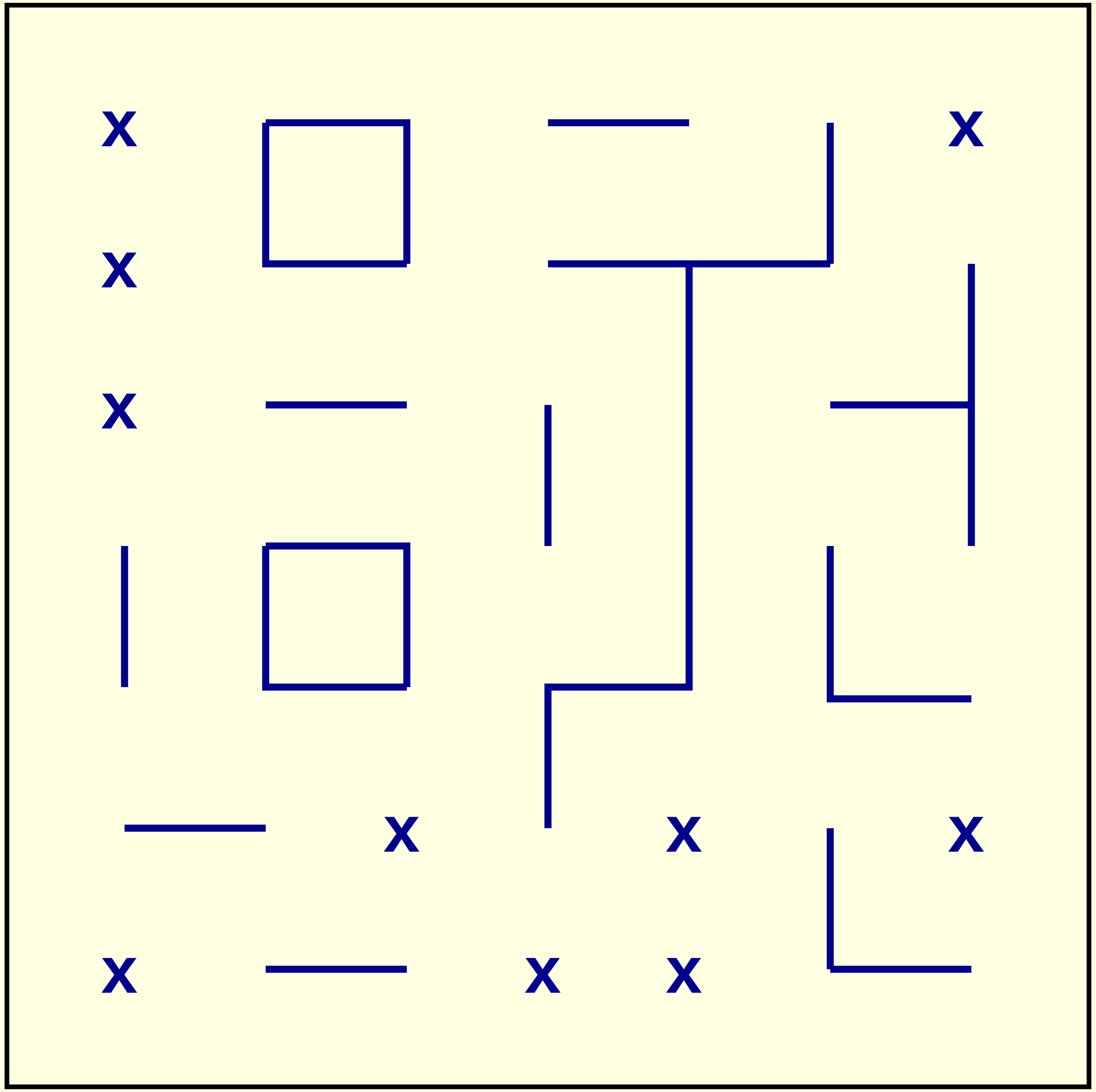
J_{ij} FM random couplings; disorder strength $\delta = \frac{variance}{mean}$ of the disorder distribution


q=3
0
1
2

$c(G) = 10 + 12 = 22$



$$c(G) \qquad \prod_{e} (e^{\beta J_e} - 1)$$



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in $2D$: $q_c = 4$ (exact result)

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Systems with Disorder

Continuous PT

HARRIS criterion

$\alpha_P > 0$ disorder is **relevant**

$\alpha_P < 0$ disorder is **irrelevant**

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will SOFTEN the transition

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$\alpha_P < 0$ disorder is **irrelevant**	will SOFTEN the transition

Relevant disorder

- Conventional Random FP (different values of critical exponents)
- Infinite Randomness FP (IRFP) (disorder grows without limits)

$q \longrightarrow \infty$ limit

$$T \rightarrow T' = T \ln q \qquad\qquad f(T) \rightarrow \frac{f(T')}{\ln q}$$

$$Z = \sum_{G \subseteq E} q^{c(G)} \prod_{ij \in G} \left[q^{\beta J_{ij}} - 1 \right]$$

$$\Downarrow_{q \rightarrow \infty}$$

$$Z = \sum_{G \subseteq E} q^{\phi(G)} \qquad\qquad \phi(G) = c(G) + \beta \sum_{ij \in G} J_{ij}$$

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$Z = n_0 q^{\phi^*} (1 + \ldots)$ where $\phi^* = \max_G \phi(G)$ and $\phi^* = -\beta N f$

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  - $\rightarrow$ ξ, correlation lenght, is the average size of the clusters

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- theorem of discrete math $\Rightarrow$ $\exists$ a combinatorial optimization method to maximize it in polynomial time
- for $\phi(G)$ of the Potts model a specific algorithm has been formulated

  Angles d'Auriac *et al.*JPA**35**, 6973 (2002)

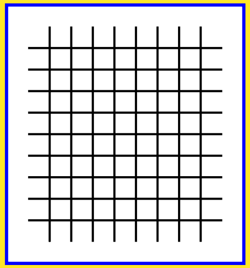
Clean case: $\delta = 0$

There exist only TWO Optimal Sets:

$$T < T_c$$

$$-\beta N f = 1 + N\beta J z$$

$z = d =$ connectivity



fully connected OS

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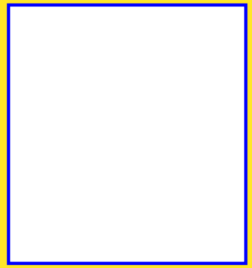

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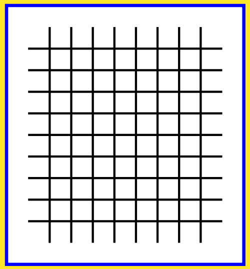

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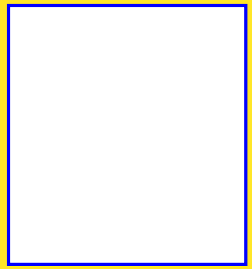

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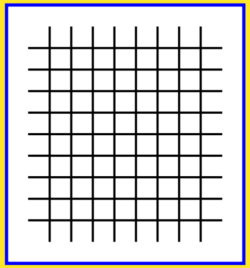

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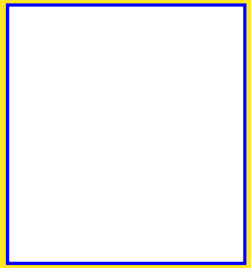

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LATENT HEAT: $\Delta e =< J > z$

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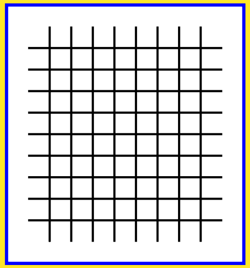

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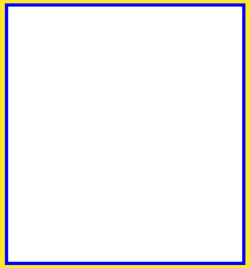

empty OS

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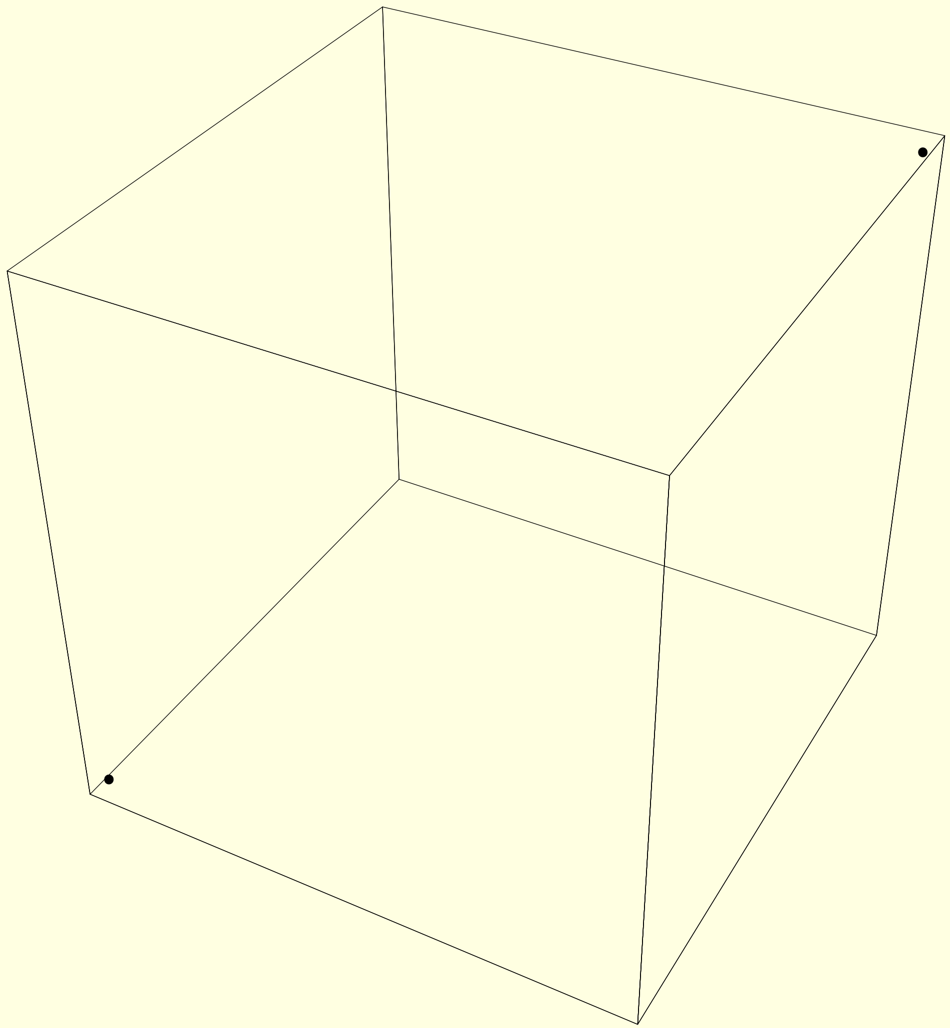
LATENT HEAT: $\Delta e =< J > z$

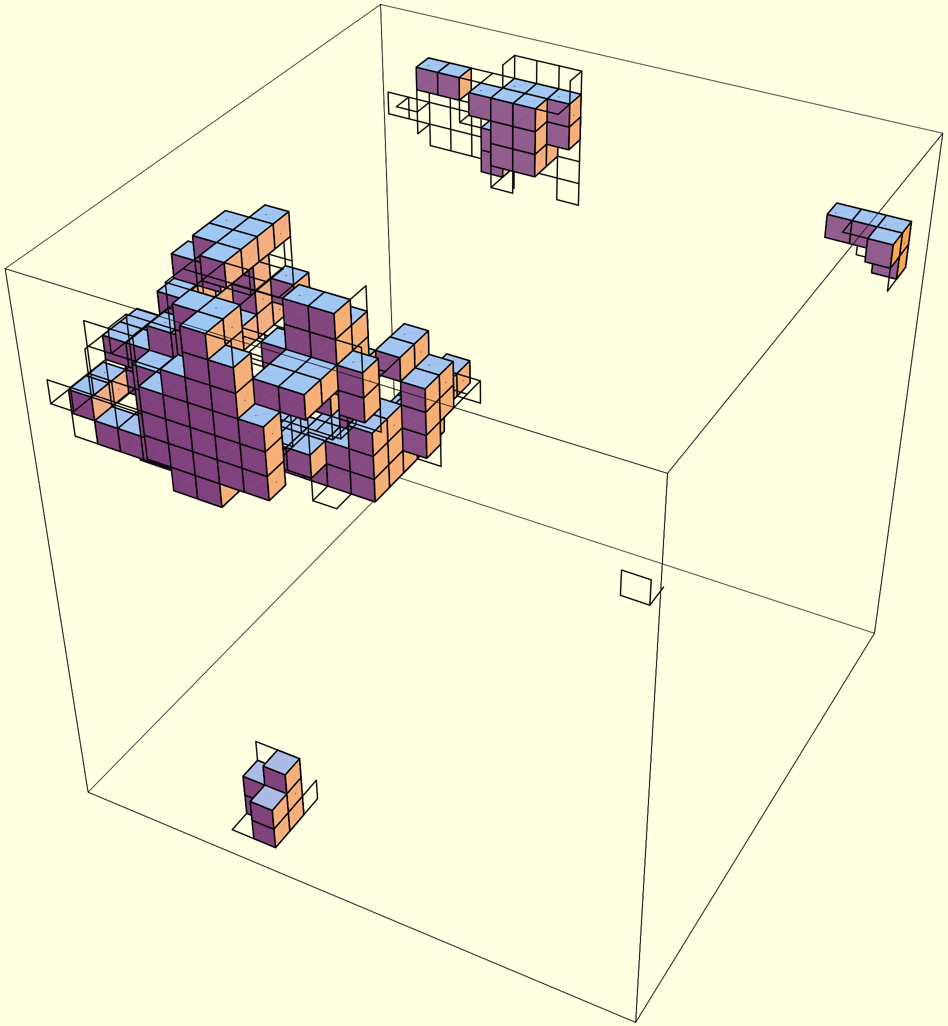
Introducing DISORDER new types of Optimal Diagrams will appear

OS at high temperature



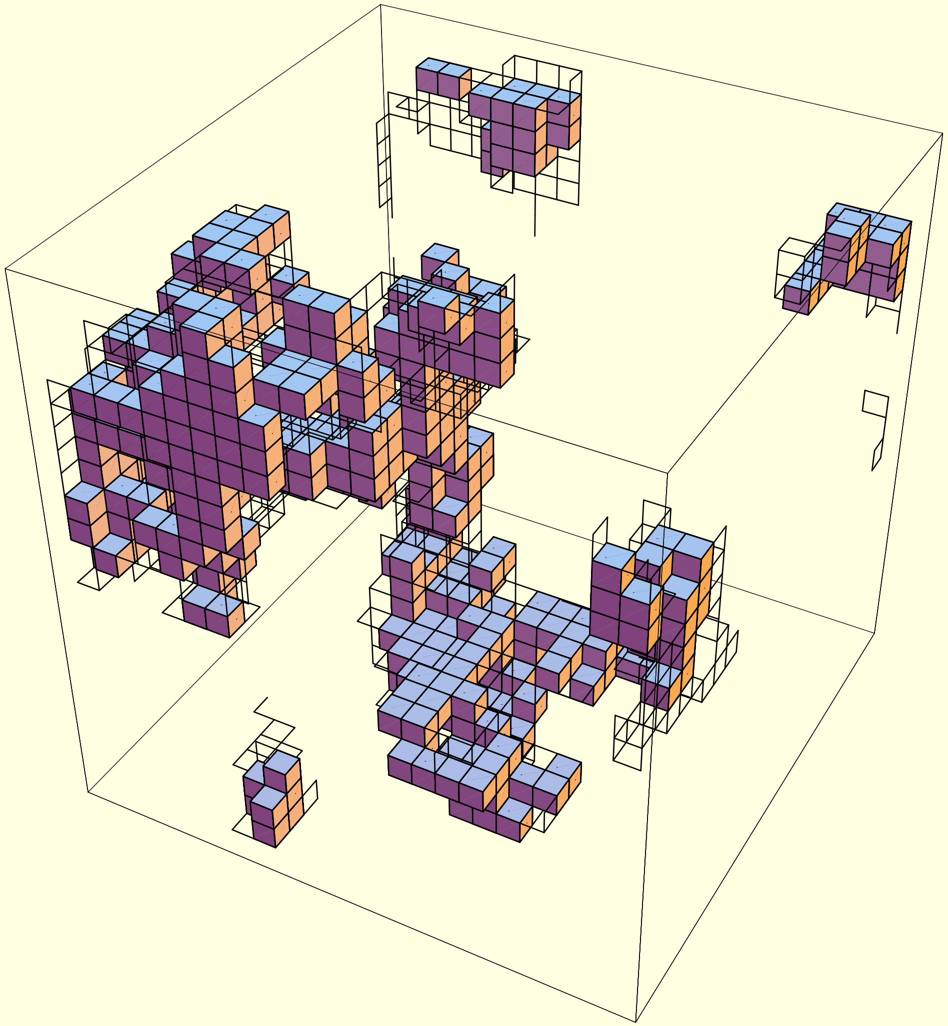
linear size $L = 24$; disorder strength $\delta = 0.875$

OS at $T = 3.2$



linear size $L = 24$; disorder strength $\delta = 0.875$

OS at $T = 3.153$



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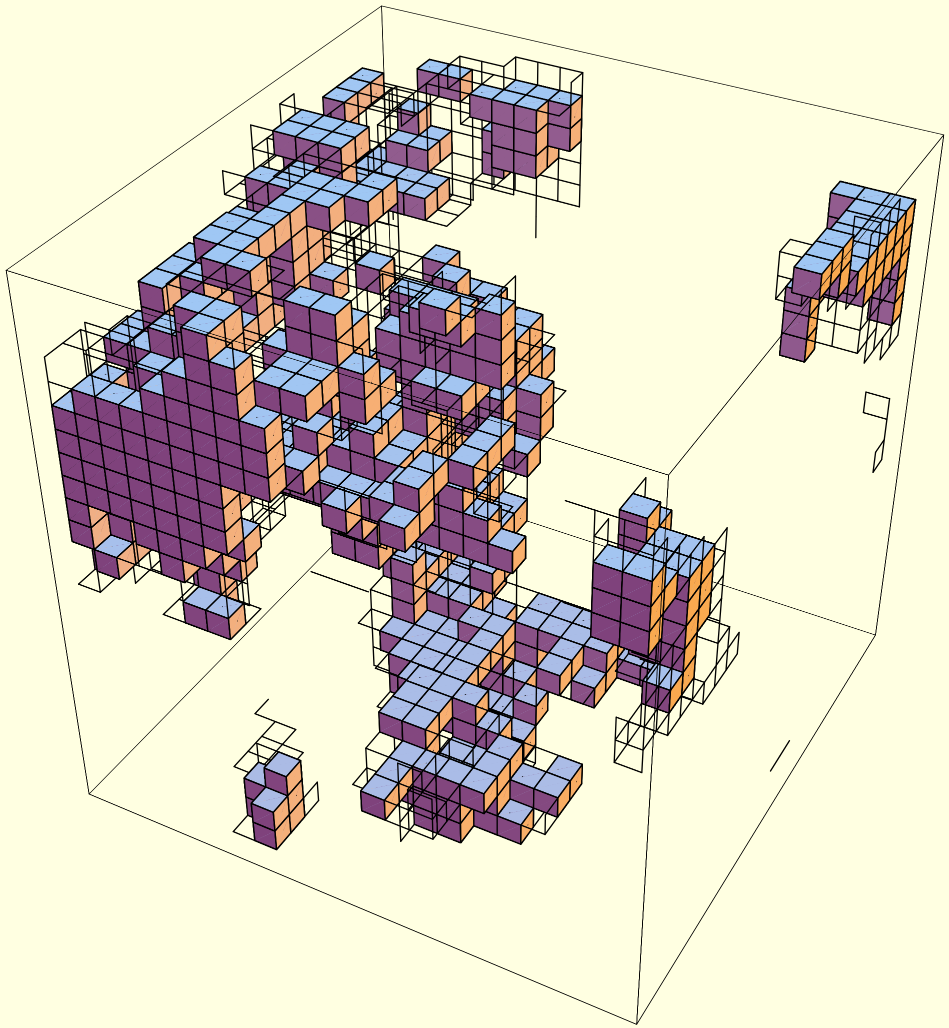
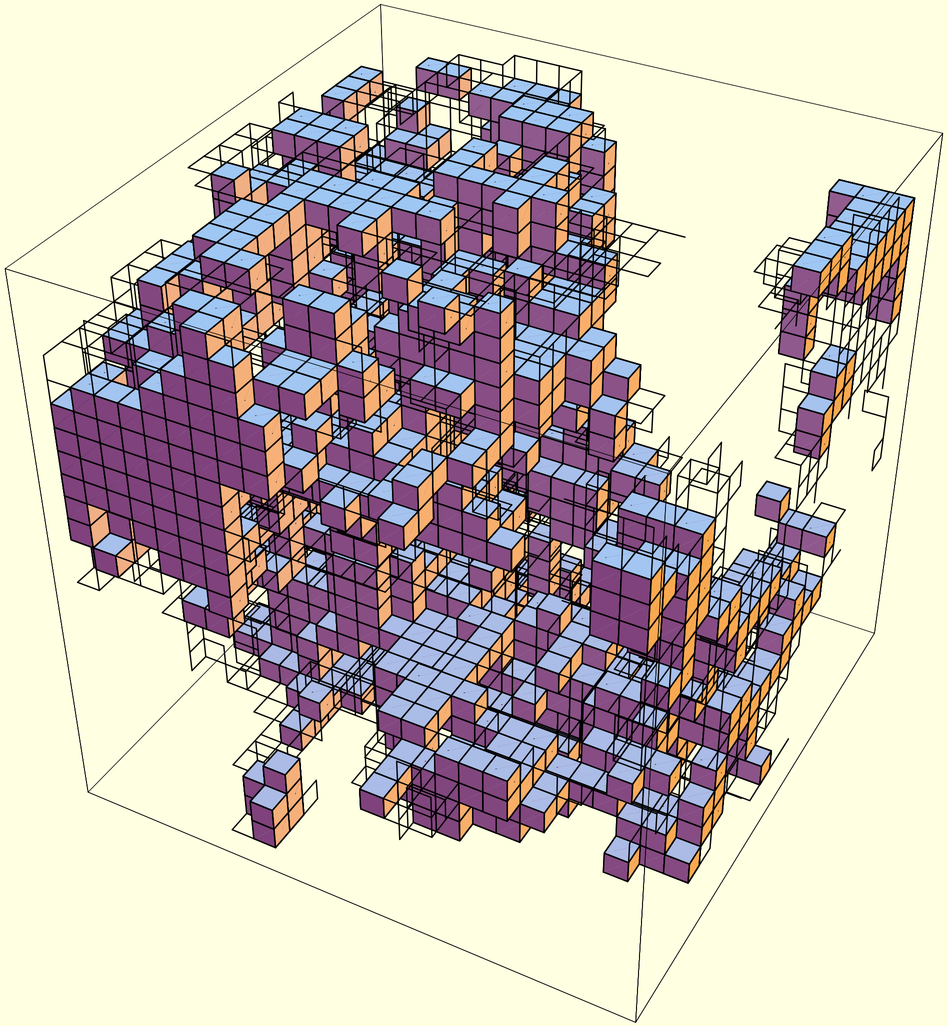
OS at $T = 3.141$

Percolating Cluster



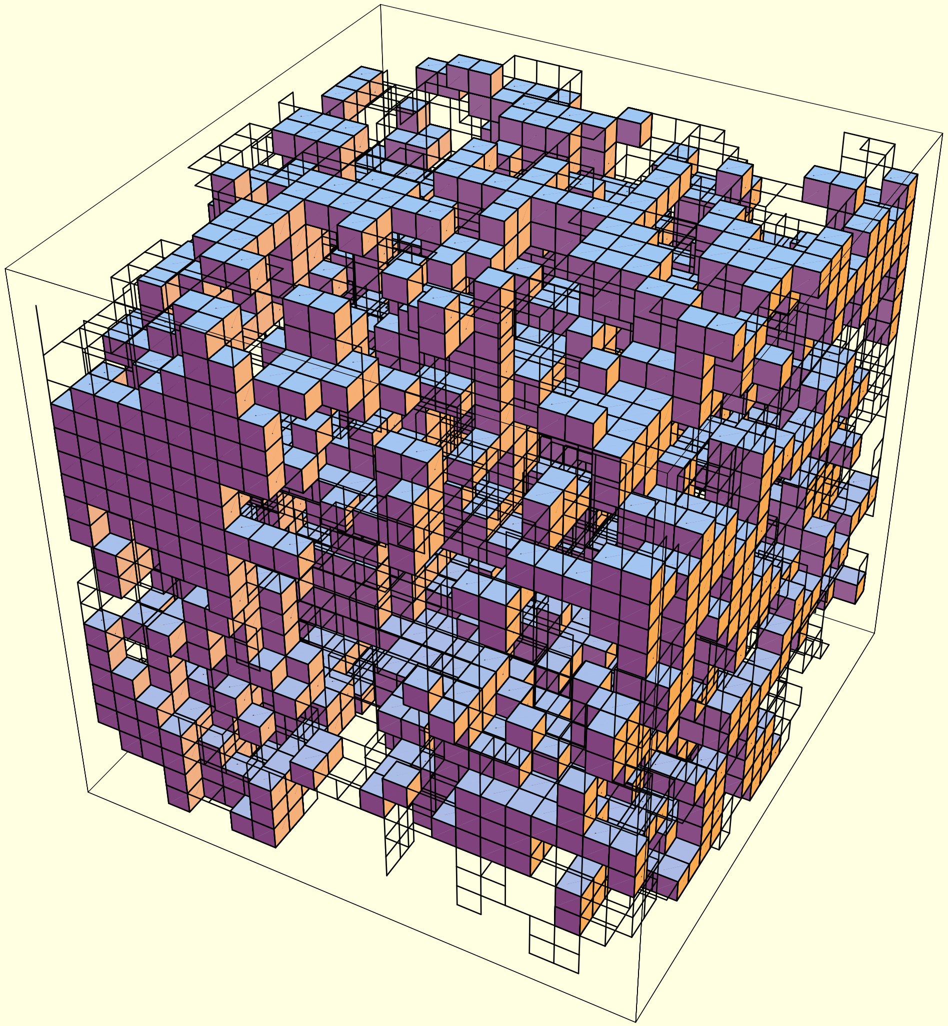
linear size $L = 24$; disorder strength $\delta = 0.875$

OS at $T = 3.128$



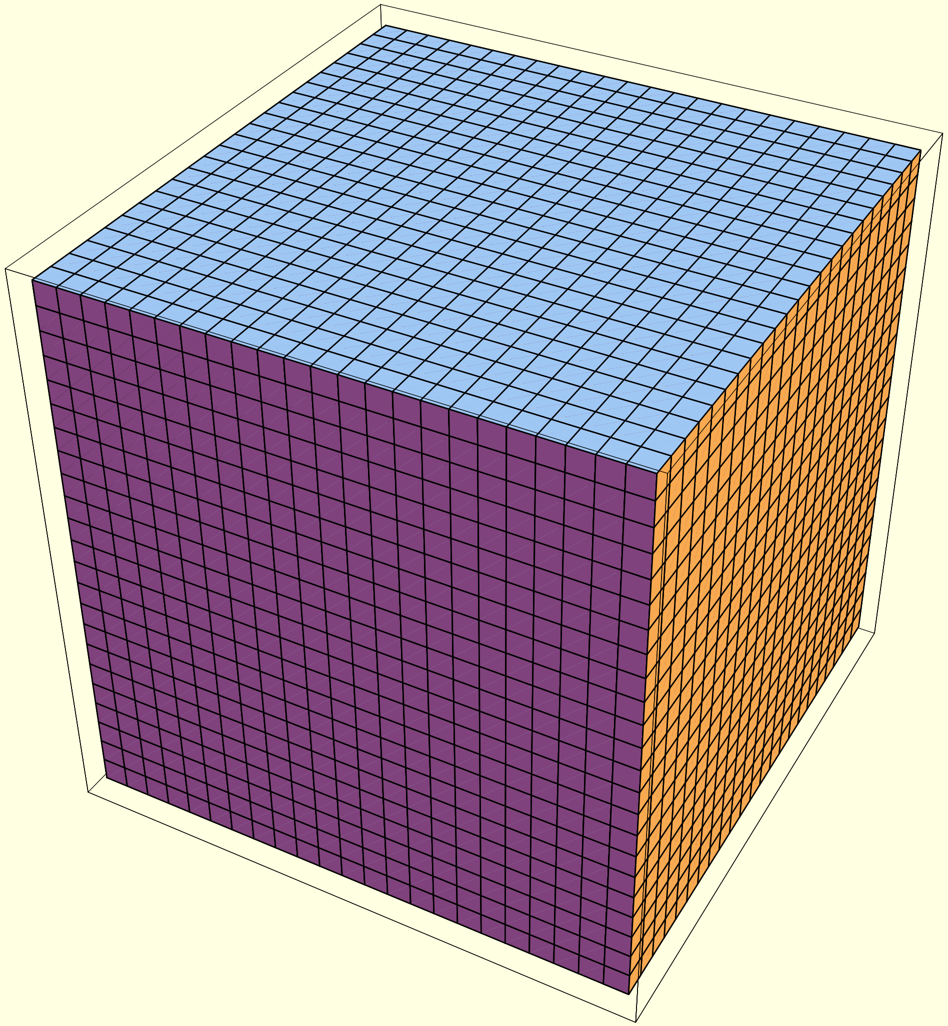
linear size $L = 24$; disorder strength $\delta = 0.875$

OS at $T = 3.122$



linear size $L = 24$; disorder strength $\delta = 0.875$

OS at $T \to 0$



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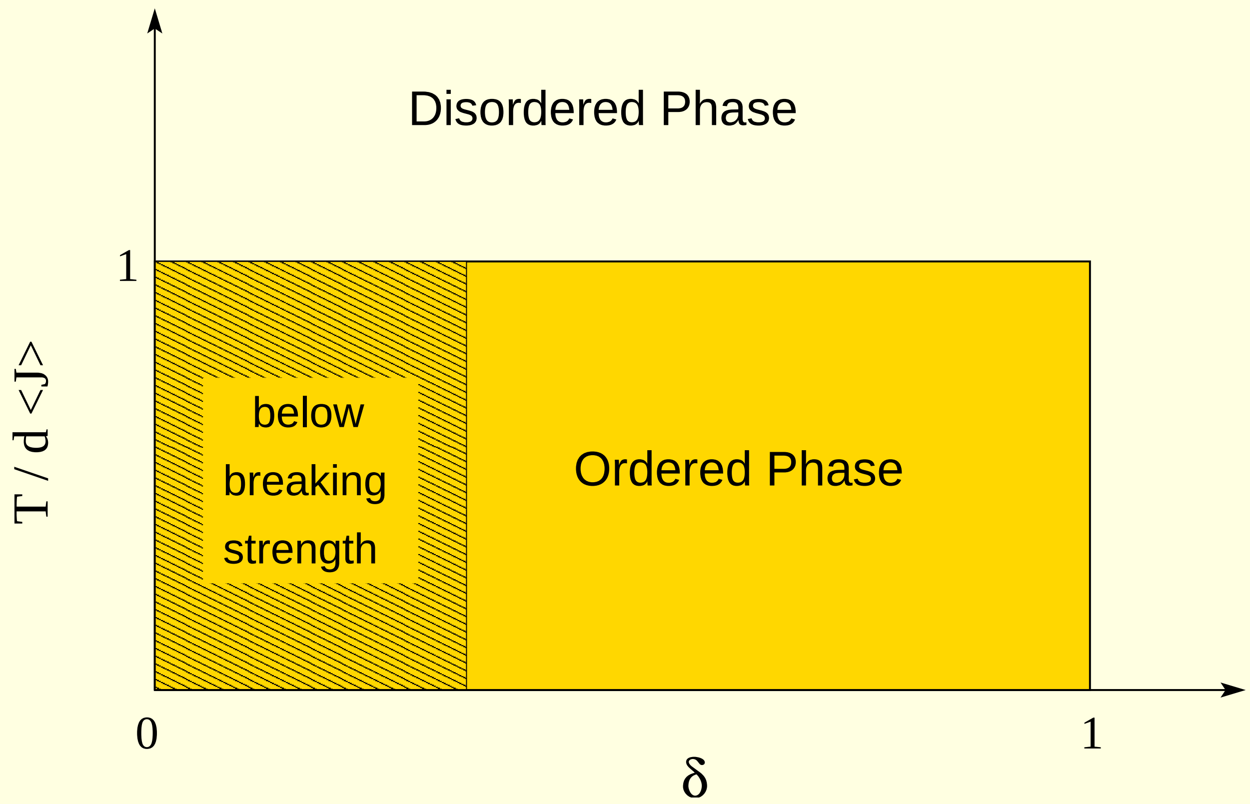
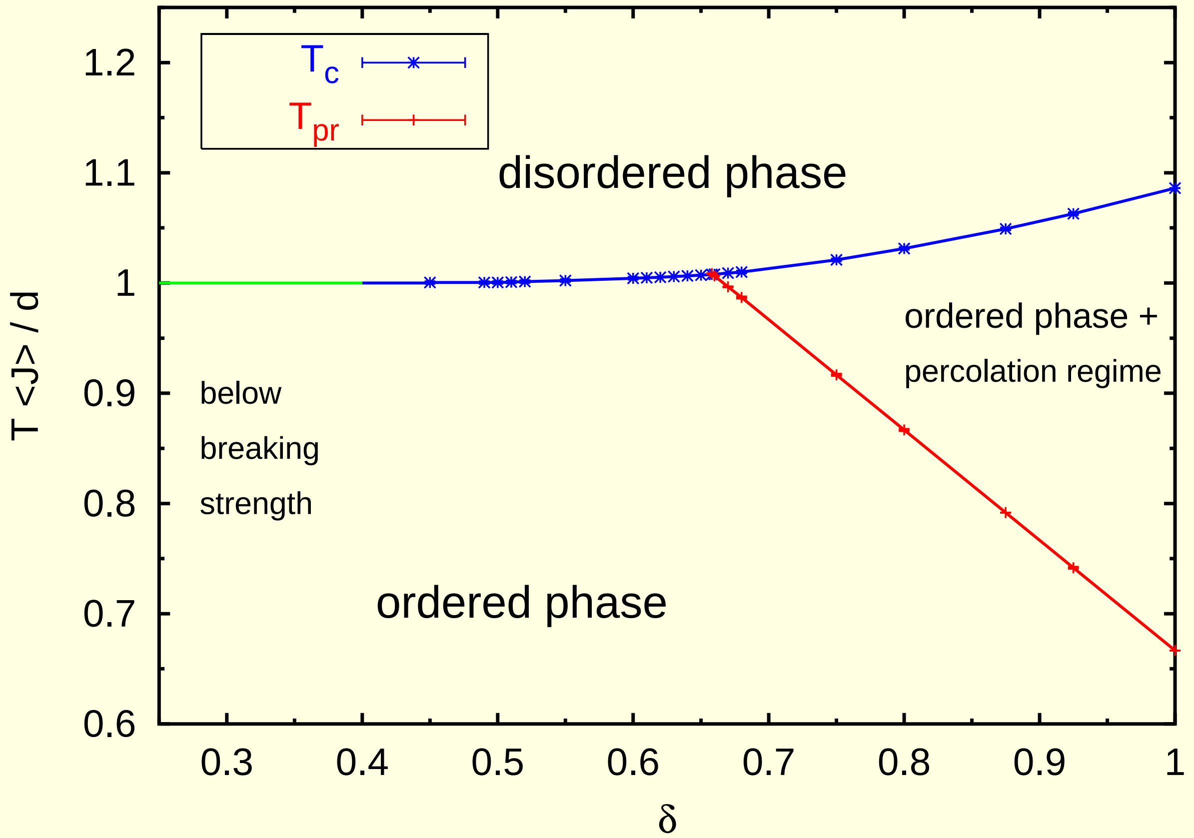
Phase Diagram - 2 D



Exact Result: $T_c = 2 < J >$ Wu, RMP **54**, 235 (1982)

In 2D Disorder destroy Phase Coexistence $\Rightarrow$ it softens the 1^{st} order PT into a 2^{nd} order PT

Phase Diagram - 3 D



Mercaldo, Anglès d'Auriac, Iglói, Europhys. Lett. **70**, 733 (2005)

T_c is not known from theory in $3D$!

IN 3D WEAK DISORDER DOES NOT DISTROY PHASE COEXISTENCE

i.e. disorder has to be strong enough to soften the PT into 2^{nd} order PT

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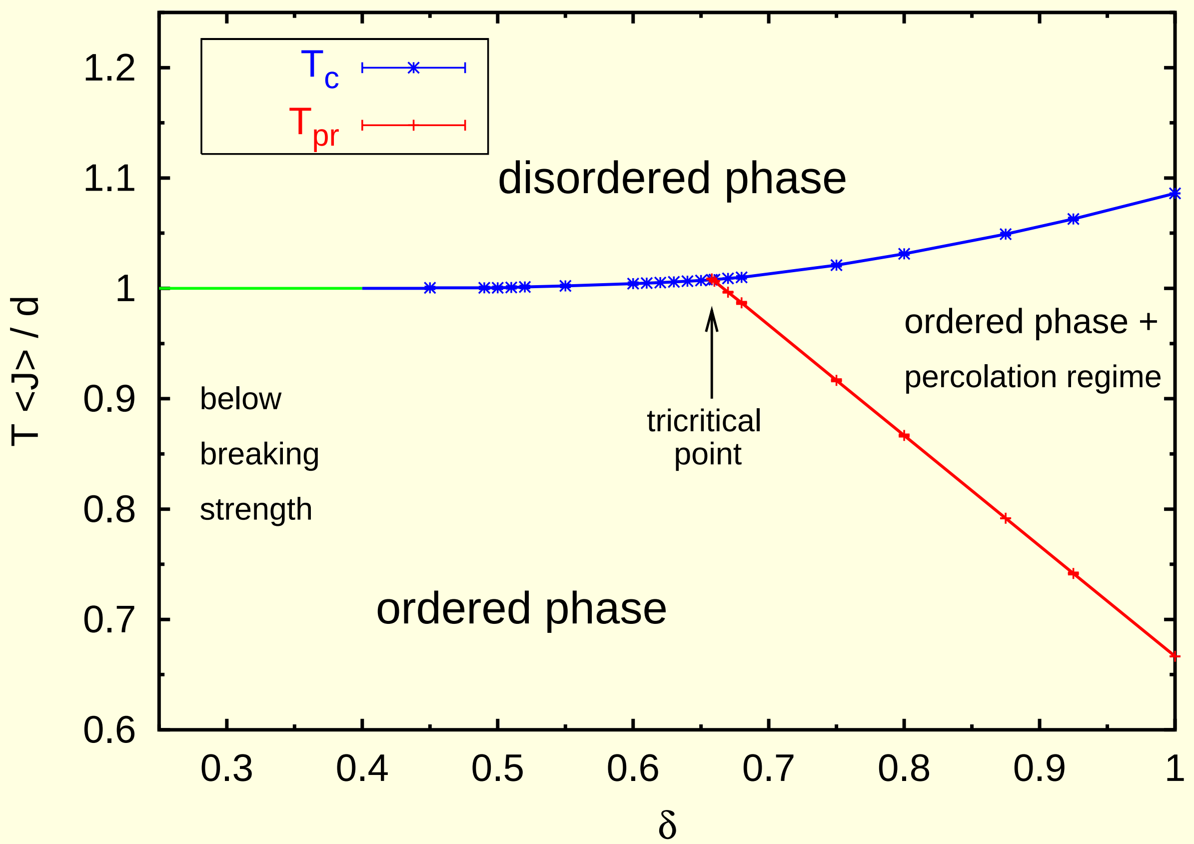

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T_c is not known from theory in $3D$!

In 3D Weak Disorder does not distroy Phase Coexistence

i.e. disorder has to be strong enough to soften the PT into 2^{nd} order PT $\Rightarrow \delta_{tr} = 0.658 \pm 0.002$

Magnetization and fractal properties of the percolating cluster

$$d_f = d - \frac{\beta}{\nu}$$

At the critical point the largest cluster of G^* is a fractal and its mass $M \sim L^{d_f}$

According to scaling theory, cumulative distribution of the mass of the cluster

$$R(M, L) = M^{-\tau} \tilde{R}(M/L^{d_f})$$

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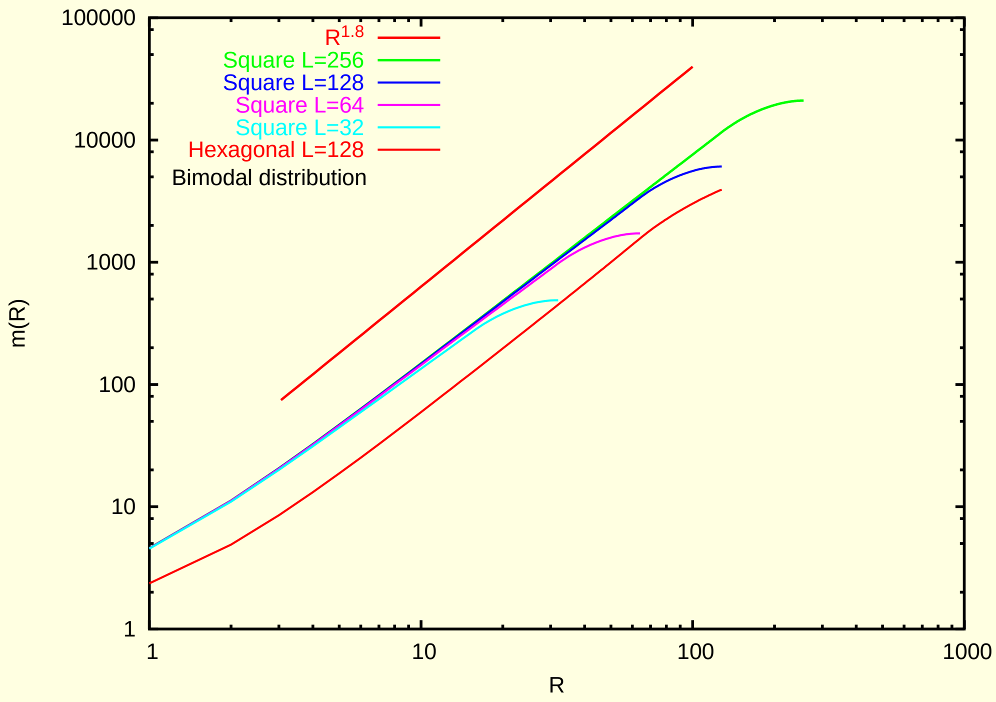
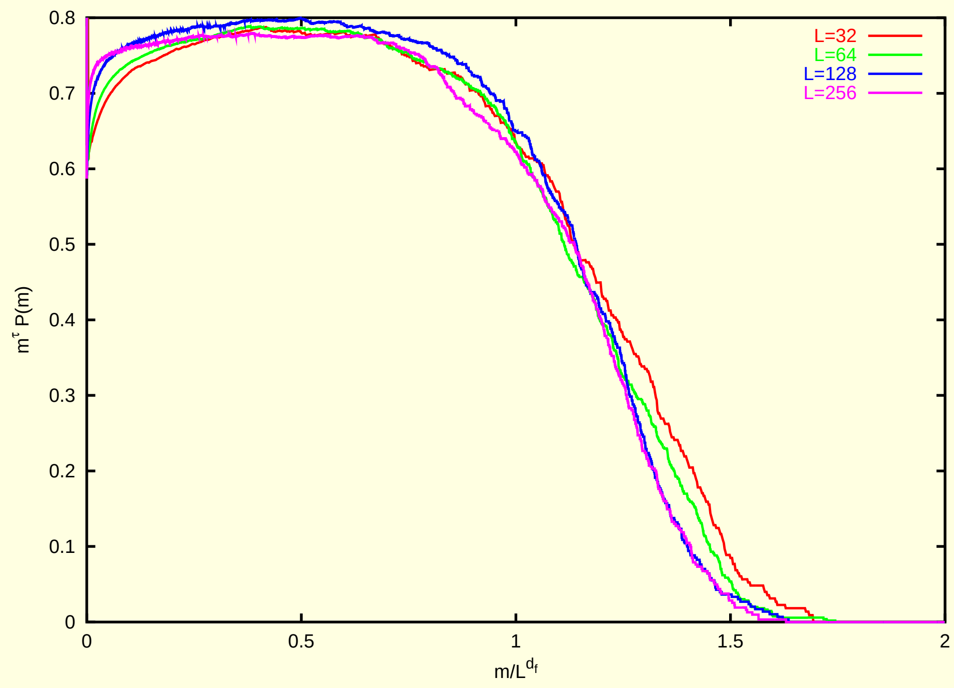
2 D: $d_f = (5+\sqrt{5})/4$





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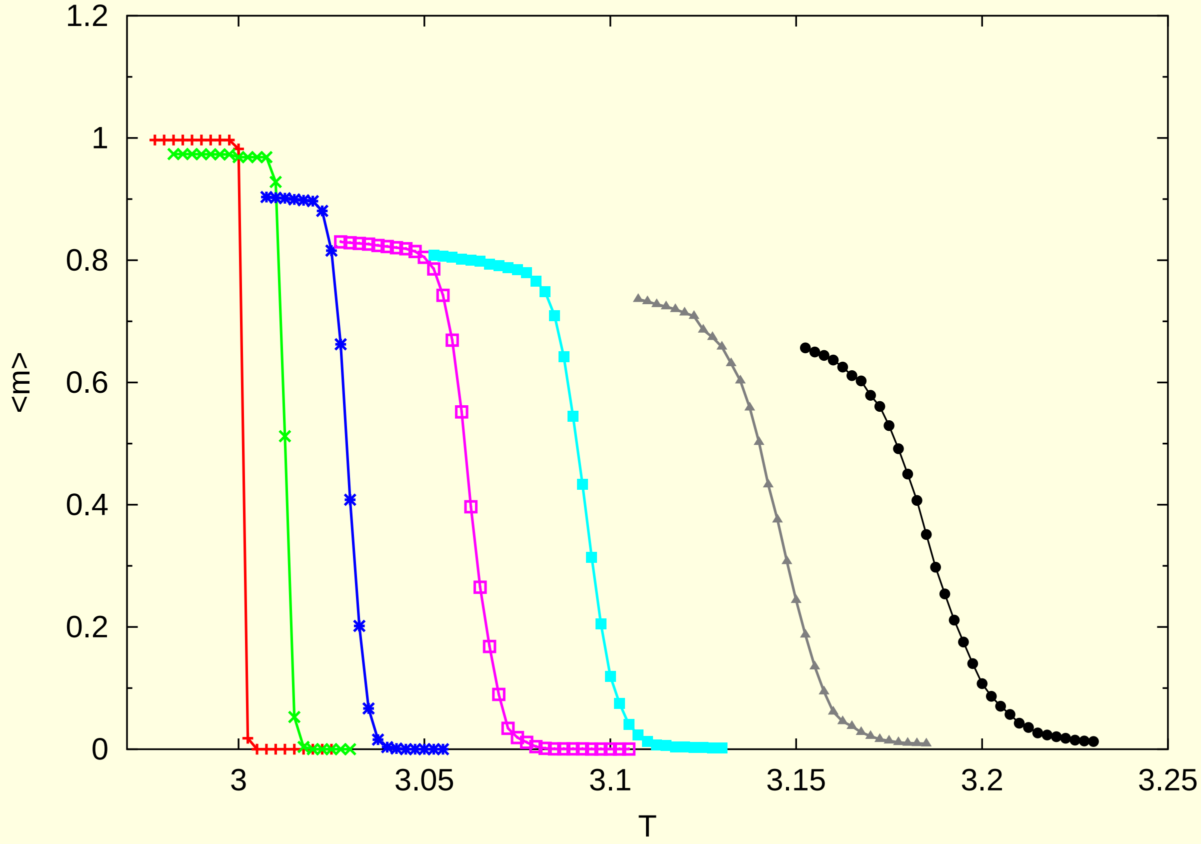

Curves of magnetization in 3 D varying the strength of disorder

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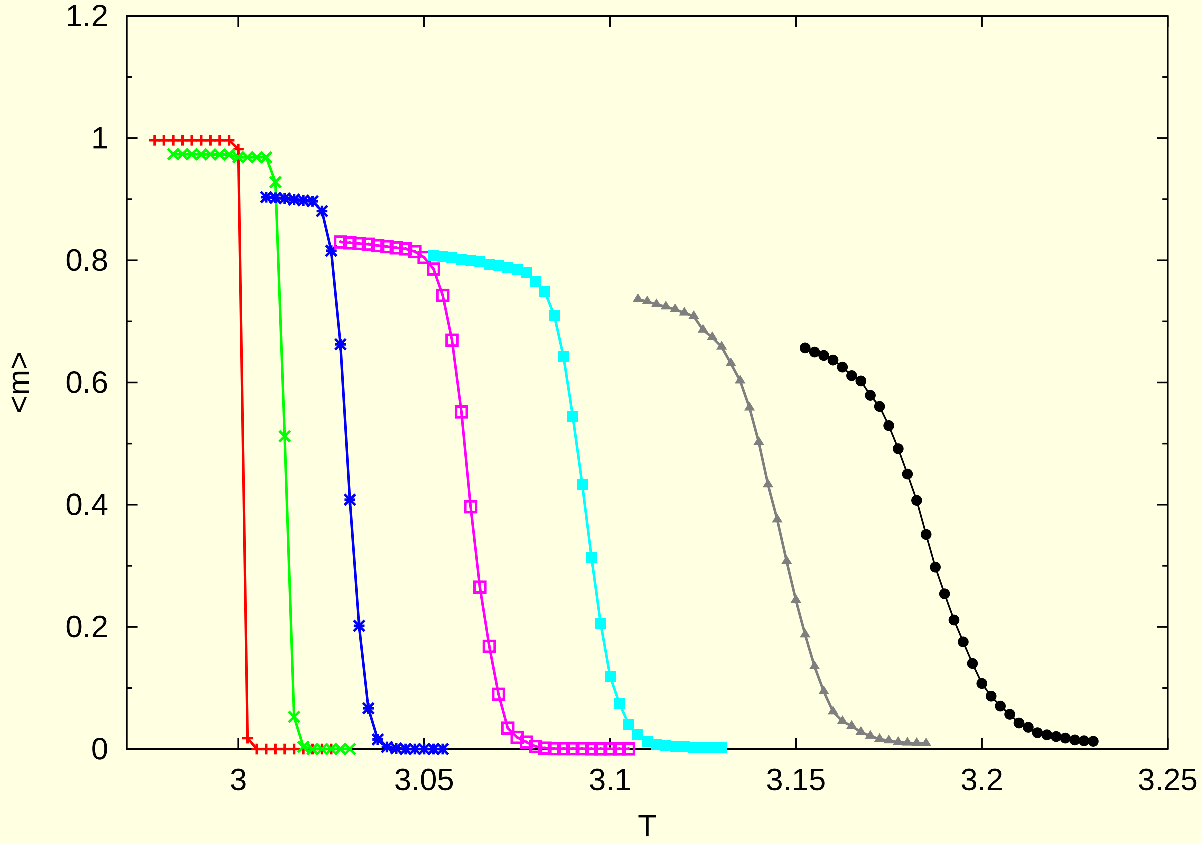

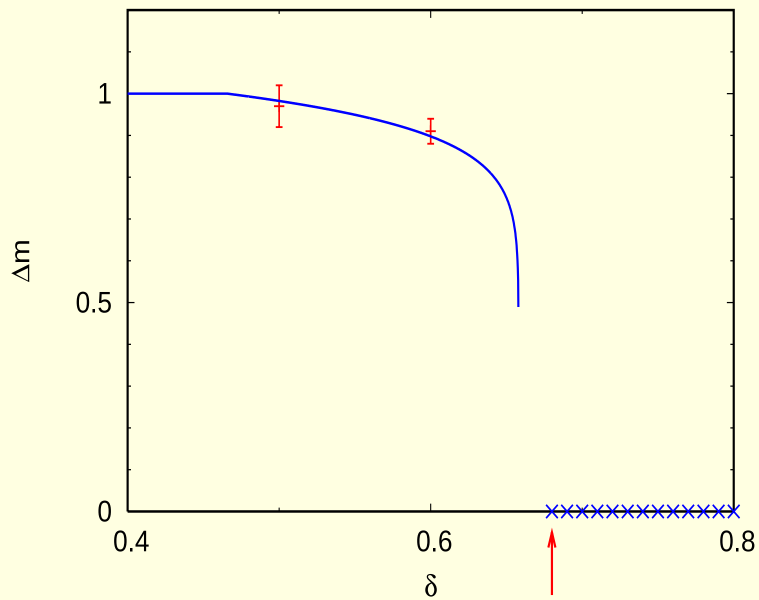

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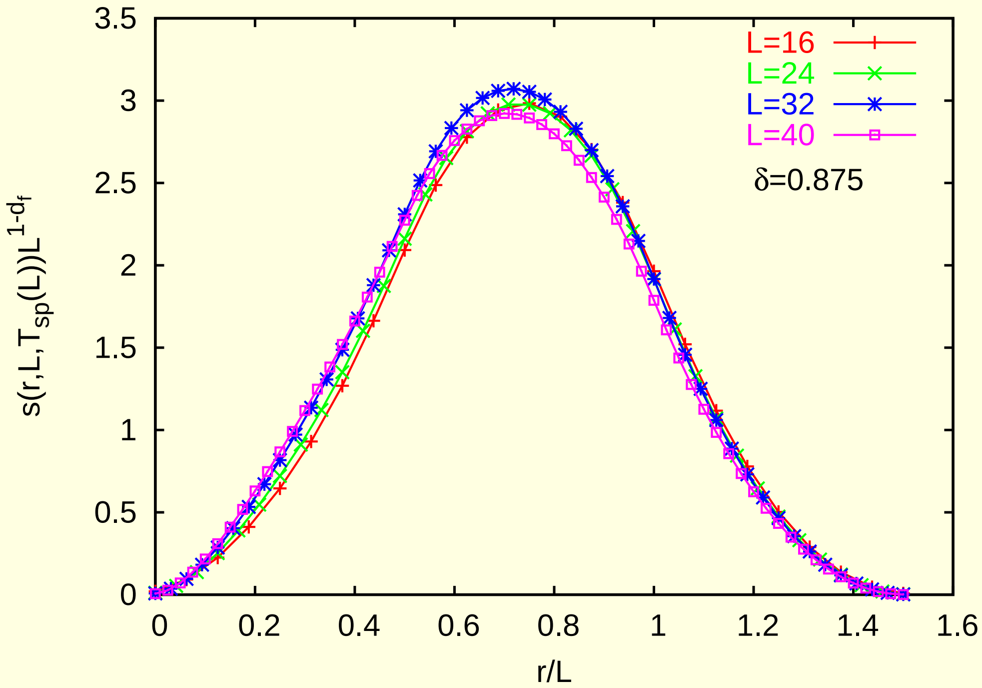

Second-order transition regime:

d_f is estimated measuring the average number of sites

belonging to the percolating cluster, $s(r, L, T)$,

in a shell of width $= 1$ and radius $= r$

$$s(r, L, t) = L^{d_f - 1} \tilde{s}(r/L, tL^{1/\nu}) \qquad t = \frac{T - T_c}{T_c}$$

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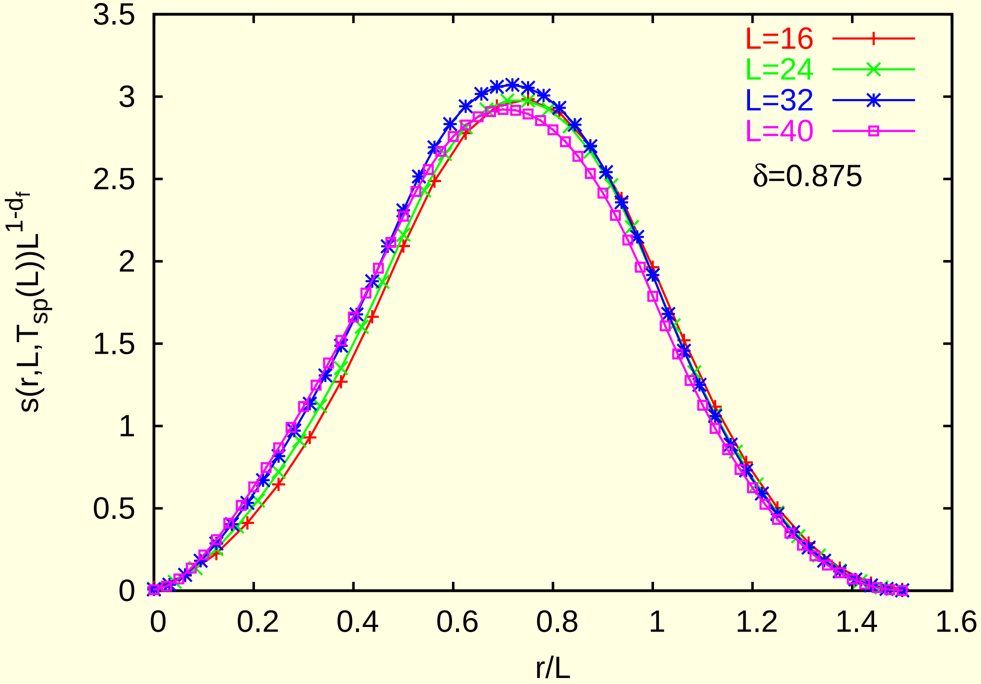

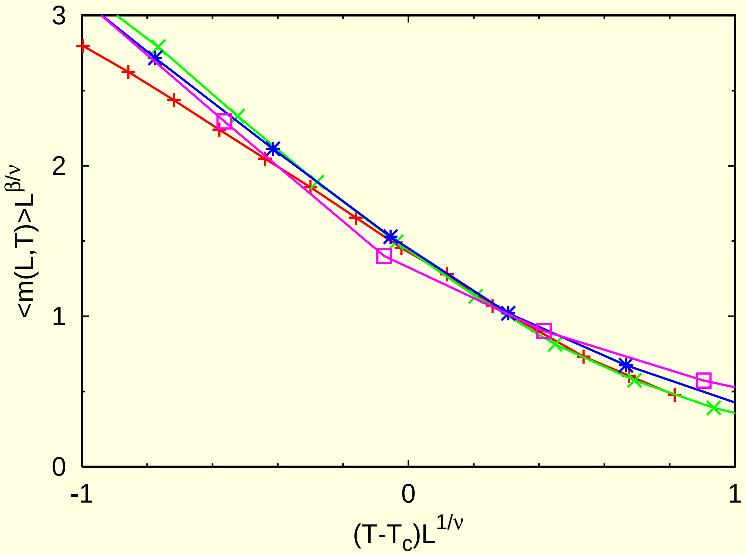

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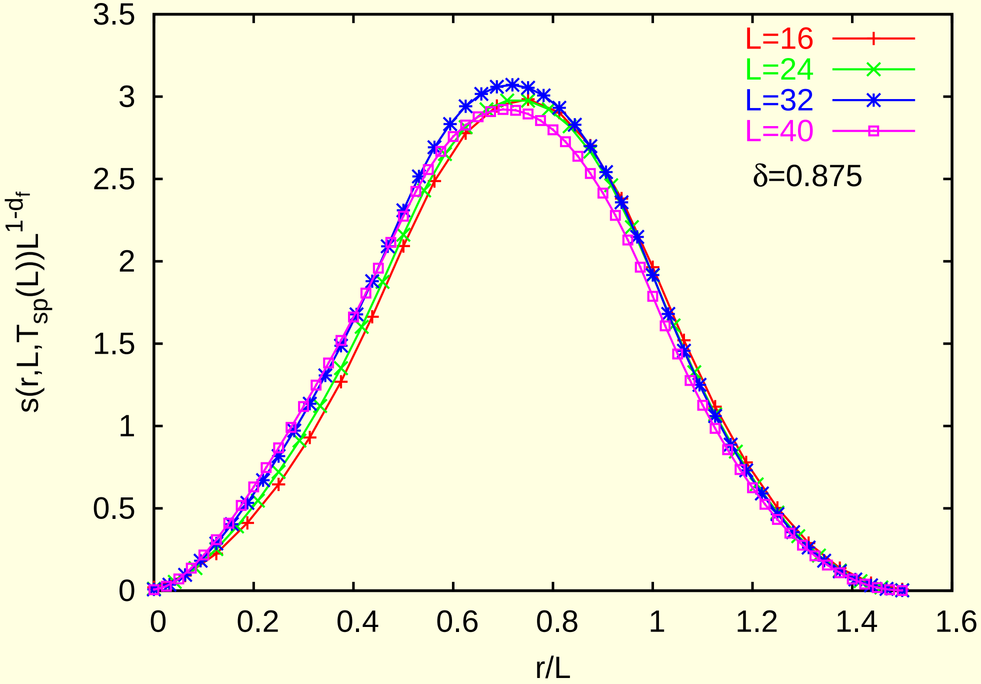

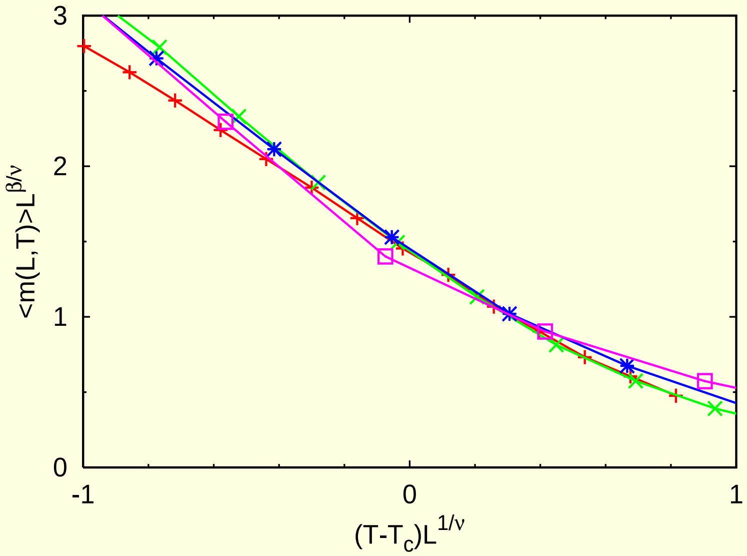

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$$d_f = 2.40 \pm 0.02$$

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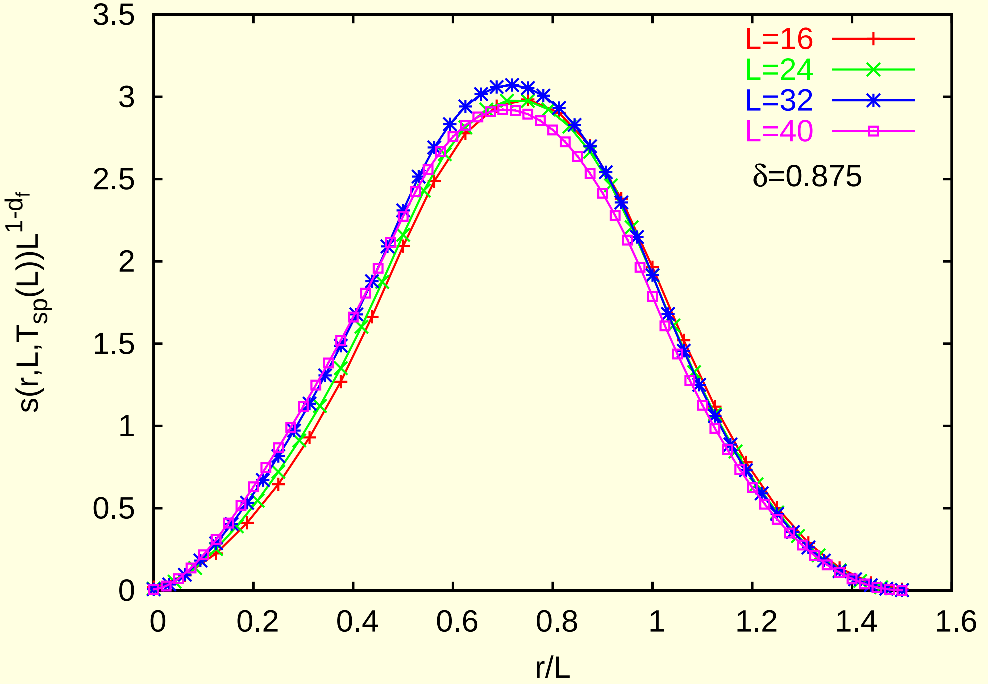

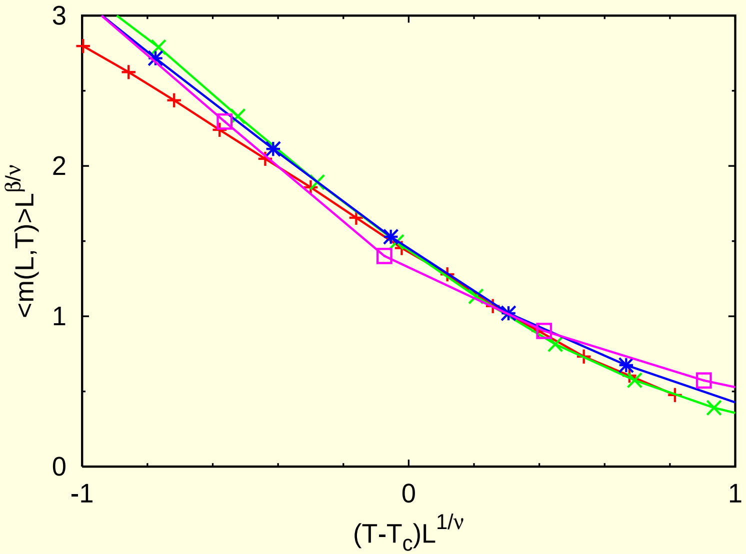

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$$\nu = 0.73 \pm 0.01$$

Conclusions

- We have studied the critical properties of the RBPM with and effi cient new algorithm, which allows to calculate EXACTLY the free energy of the system

- We have analyzed thermal and magnetic properties of different kind of lattices, with different distribution of disorder in **2D**, and also for cubic lattice with bimodal distribution of disorder in **3D**